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Vector bundle moduli superpotentials in heterotic superstrings and M-theory

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Abstract: The non-perturbative superpotential generated by a heterotic superstring wrapped once around a genus-zero holomorphic curve is proportional to the pfaffian involving the determinant of a Dirac operator on this curve. We show that the space of zero modes of this Dirac operator is the kernel of a linear mapping that is dependent on the associated vector bundle moduli. By explicitly computing the determinant of this map, one can deduce whether or not the dimension of the space of zero modes vanishes. It is shown that this information is sufficient to completely determine the pfaffian and, hence, the non-perturbative superpotential as explicit holomorphic functions of the vector bundle moduli. This method is illustrated by a number of non-trivial examples.

Keywords: Nonperturbative Effects, Superstrings and Heterotic Strings, M-Theory.

Contents

1. Introduction

The non-perturbative superpotentials for the moduli of superstrings and M-theory have been calculated using string worldsheet conformal field theory [1, 2] and, alternatively, by considering the contributions to the partition function of strings wrapped on holomorphic curves z in the compactification space [3, 4]. This latter approach has been used to compute non-perturbative superpotentials in both weakly [5] and strongly [6, 7] coupled heterotic superstring theories on a Calabi-Yau threefold, as well as in other theories [8, 9]. The superpotentials in heterotic string theory were found to be proportional to a factor involving the Wess-Zumino-Witten term, which couples the superstring to the background SO(32) and $E_8 \times E_8$ gauge bundle V [5, 6, 7]. Assuming a mild condition on the structure group, this factor can be shown to be the pfaffian of a chiral Dirac operator twisted by the restriction, $V|_z$, of the gauge bundle to the holomorphic curve. In [5], it was shown that this pfaffian and, hence, the superpotential will vanish if and only if $V|_z$ is non-trivial. Furthermore, it is clear that the pfaffian must be a holomorphic function of gauge bundle moduli associated with $V|_z$. However, until recently, neither the vanishing locus of the pfaffian in vector bundle moduli space nor its explicit dependence on these moduli was known. This was rectified in [10], where we provided solutions to both of these problems within the framework of weakly and strongly coupled $E_8 \times E_8$ theories compactified on elliptically fibered Calabi-Yau threefolds. In [10], we emphasized concepts, and did not discuss details of the requisite mathematics. Furthermore, we applied our method to a single, non-trivial example. It is the purpose of this paper to greately expand on the content of [10]. We will present the complete mathematical structure and then, using this structure, compute many examples of non-perturbative superpotentials.

Specifically, in this paper we do the following. In section 2, we review the computation of non-perturbative superpotentials in both weakly and strongly coupled heterotic string theory following the approach in [3, 4]. The superpotentials are shown to be proportional to the pfaffian of a chiral Dirac operator, $\mathcal{D}_-$, twisted by the vector bundle restricted to the holomorphic curve on which the string is wrapped. In this paper, we will compute $\mathrm{Pfaff}(\mathcal{D}_-)$ algebraically. However, in appendix A, an alternative form for the pfaffian, written in terms of an explicit gauge one-form solution of the hermitean Yang-Mills equations is derived. We discuss how non-perturbative superpotentials might be computed using this formalism. In section 3, the moduli of stable, SU(n) holomorphic vector bundles, V, over elliptically fibered Calabi-Yau threefolds, X, are enumerated and their properties are briefly discussed. Relevant information about such threefolds, and the construction of holomorphic vector bundles from spectral data is presented in appendix B. We then derive an explicit formula for the number of vector bundle moduli when the base space of X is any Hirzebruch surface, that is, $B = \mathbb{F}_r$. These results were previously obtained in [11]. Next, a holomorphic curve z is specified and the restriction of the vector bundle to the surface $\pi^* z \subset X$ is discussed. We describe and enumerate the moduli associated with $V|_{\pi^* z}$, giving an explicit formula for the number of such moduli when $B = \mathbb{F}_r$ and $z = \mathcal{S}$. Throughout this paper, when specificity is required, we will restrict $r = 0, 1, 2$. The global structure of the moduli space is also discussed. The first part of section 4 is devoted to presenting these restricted bundle moduli from the point of view of the curve z. It is shown that these moduli appear as sections of a specific rank n vector bundle over z. This requires expressions for the image on z of certain line bundles on $\pi^* z$. These expressions are derived in appendix C. In the second part of this section, we compute the restriction of V to z and show that it is the image of the spectral data line bundle. Thus, on curve z, we completely categorize $V|_z$ and the associated moduli. The main results of this paper are calculated in sections 5 and 6.

In section 5, we first discuss Pfaff$(\mathcal{D}_-)$ and show that it will vanish if and only if the dimension of the space of holomorphic sections $h^0(z, V|_z \otimes S_-)$, where S_- is the negative chirality spin bundle on z, is non-zero. It is then demonstrated that $h^0(z, V|_z \otimes S_-)$ will be non-zero, for all vector bundle moduli, for a specific set of vector bundles. We then present large classes of theories with $z = \mathcal{S}$ in which, generically, $h^0(z, V|_z \otimes S_-)$ can vanish over a subset of moduli space. In these examples, the associated cohomology class $H^0(z, V|_z \otimes S_-)$ is contained in an exact sequence of the form

$$0 \to H^0(z, V|_z \otimes S_-) \to W_1 \xrightarrow{f} W_2 \to \dots . \tag{1.1}$$

Here, W_1 and W_2 are linear spaces of identical finite dimension and, hence, linear map f can be represented by a square matrix. Furthermore, f is an explicit function of the vector bundle moduli associated with $V|_z$. It follows that $h^0(z, V|_z \otimes S_-)$ will be non-zero if and only if the determinant of f vanishes. Thus, the pfaffian and, hence, the superpotential will vanish if and only if $\det f = 0$. In section 6, we compute the matrix f and its determinant for a number of explicit examples, all, for simplicity, with $B = \mathbb{F}_r, z = \mathcal{S}$ and structure group $G = \mathrm{SU}(3)$. To begin, we present an example of the type where, although matrix f is non-trivial, $\det f$ vanishes for all values of the vector bundle moduli. Therefore, for this example, Pfaff$(\mathcal{D}_-)$ and the superpotential always vanish. However, we then present three explicit examples where $\det f$ are explicit, non-zero functions of the vector bundle moduli. In one example, a polynomial appearing as a factor in $\det f$ is very lengthy and, hence, is presented in appendix D. In these examples $\det f$ may vanish, but only on a divisor of the moduli space. Using $\det f$, one can, in each case, explicitly compute the zero locus of the pfaffian. Away from the zero locus, Pfaff$(\mathcal{D}_-)$ and, hence, the superpotential are non-vanishing. Finally, in section 7, we show how the pfaffian can be explicitly computed from the knowledge of its zero locus. We find that Pfaff$(\mathcal{D}_-) \propto \det f$ and, therefore, that the non-perturbative superpotential W satisfies $W \propto \det f$. Since $\det f$ was explicitly computed as a function of the associated vector bundle moduli in section 6, we have determined the vector bundle moduli contribution to the superpotential.

Although at first sight rather complicated to derive, the superpotential for vector bundle moduli potentially has a number of important physical applications. To begin with, it is essential to the study of the stability of the vacuum structure [12, 13] of both weakly coupled heterotic string theory and heterotic M-theory [14]–[21]. Furthermore, in both theories it allows, for the first time, a discussion of the dynamics of the gauge bundles. For example, in heterotic M-theory one can determine if a bundle is stable or whether it decays, via a small instanton transition [22], into five-branes. In recent years, there has been considerable research into the cosmology of superstrings and heterotic M-theory [23, 24, 25]. In particular, a completely new approach to early universe cosmology, Ekpyrotic theory [26]–[30], has been introduced within the context of brane universe theories. The vector bundle superpotentials discussed in this paper and [10] allow one to study the dynamics of the small instanton phase transitions that occur when a five-brane [26, 27] or an "end-of-the-world" orbifold plane [28, 29, 30] collides with our observable brane, thus producing the Big Bang. These physical applications will be discussed elsewhere.

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2. Non-perturbative superpotentials

Let us consider the long wavelength limit of $E_8 \times E_8$ heterotic superstring theory compactified on a Calabi-Yau threefold, X. This four-dimensional, $N = 1$ supersymmetric effective theory contains a number of chiral supermultiplets, some charged with respect to the low-energy gauge group and some uncharged. Of the uncharged supermultiplets, a subset corresponds to the moduli of the compactification. In heterotic string theory, these moduli break into two types, Calabi-Yau moduli corresponding to the Kahler and complex structure deformations of X and the vector bundle moduli. These latter fields are associated with the integration constants in the vacuum solutions of the hermitean Yang-Mills equations for the gauge connection on a fixed X. Here, we will simply denote any $N = 1$ chiral moduli superfield by

$$(Y^{I'}, \lambda^{I'}), \tag{2.1}$$

where $Y^{I'}$ is the complex scalar and $\lambda^{I'}$ is its spin $\frac{1}{2}$ superpartner. The index I' runs over all Calabi-Yau and vector bundle moduli. No matter what their origin, the dynamics of these moduli is given by a lagrangian of the form

$$\begin{aligned}\mathcal{L}_{4D} = {} & K_{I'\bar{J}'}\partial_u Y^{I'}\partial^u \bar{Y}^{\bar{J}'} + e^{\kappa_p^2 K}\left(K^{I'\bar{J}'}D_{I'}W\bar{D}_{\bar{J}'}\bar{W} - 3\kappa_p^2|W|^2\right) + \\ & + K_{I'\bar{J}'}\lambda^{I'}\partial\!\!\!/\bar{\lambda}^{\bar{J}'} - e^{\frac{\kappa_p^2 K}{2}}(D_{I'}D_{J'}W)\lambda^{I'}\lambda^{J'} + \text{h.c.} + \cdots\end{aligned} \tag{2.2}$$

Here κ_p^2 is the four-dimensional Newton's constant,

$$K_{I'\bar{J}'} = \partial_{I'}\partial_{\bar{J}'}K \tag{2.3}$$

are the Kahler metric and the Kahler potential respectively, and

$$D_{I'}W = \partial_{I'}W + \kappa_p^2\frac{\partial K}{\partial Y^{I'}}W \tag{2.4}$$

is the Kahler covariant derivative acting on the superpotential W. A priori, both K and W are arbitrary. However, since the supermultiplets $(Y^{I'}, \lambda^{I'})$ correspond to the moduli of the compactification of the heterotic string on X, one must have

$$W = 0\,. \tag{2.5}$$

The moduli Kahler potential, on the other hand, is non-trivial and can have a complicated functional structure on $Y^{I'}$. Fortunately, we do not need to know the explicit form of K in this paper.

Both perturbative and non-perturbative quantum processes can, in principle, induce a non-vanishing superpotential for the moduli fields. However, it is well known that perturbative radiative corrections to the superpotential vanish. However, non-zero corrections to W can be generated by non-perturbative processes involving superstring instantons. Following the approach of [3, 4], we will calculate the non-perturbative superpotential by computing instanton induced fermion bilinear interactions and then comparing these to the fermion bilinear terms in (2.2). In this paper, the instanton contribution arises from a

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heterotic superstring wrapped once around a holomorphic curve $z \subset X$. In order to isolate the vector bundle moduli contribution to the superpotential, the main objective of this paper, we find it expedient to calculate the two point functions of fermions associated with the Calabi-Yau moduli. Denote these moduli superfields by

$$(T^I, \lambda_T^I), \tag{2.6}$$

where index I runs over Calabi-Yau moduli only. Then the two-point function of fermions λ_T^I and λ_T^J, located at positions y_1 and y_2, is given by the path integral expression

$$\begin{aligned}\langle \lambda^I(y_1)\lambda^J(y_2)\rangle &\propto \int \mathcal{D}\lambda e^{-\int d^4y \sum_K \lambda^K \not{\partial} \lambda^K} \lambda^I(y_1)\lambda^J(y_2) \\ &\propto \int \mathcal{D}\mathbb{Z}\mathcal{D}g e^{-S_z(\mathbb{Z},g;\mathbb{E}_{\mathbb{M}}^{\mathbb{A}},\mathbb{B}_{\mathbb{M}\mathbb{N}},\phi,\mathbb{A}_{\mathbb{M}})},\end{aligned} \tag{2.7}$$

where here, and henceforth, we drop the subscript T in λ_T^I. $\mathbb{Z}$ and g are the worldsheet fields on the heterotic superstrings, with

$$\mathbb{Z} = (X, \Theta) \tag{2.8}$$

being the boson and fermion coordinates of the target superspace respectively, and g being the map from z to the gauge group. In addition, $\mathbb{E}_{\mathbb{M}}^{\mathbb{A}}, \mathbb{B}_{\mathbb{M}\mathbb{N}}, \phi$ and $\mathbb{A}_{\mathbb{M}}$ are the background superfield vielbein, two-form, dilaton and supergauge connection in which the superstring propagates. S_z is the heterotic string action. The result of this calculation is then compared to the term in (2.2) proportional to $(D_I D_J W)\lambda^I \lambda^J$, and the non-perturbative contribution to W extracted. Note that W is non-perturbative from the point of view of the string worldsheet theory and, hence, must be proportional to $e^{-1/\alpha'}$, where α' is the worldsheet constant. However, it does not contain the dilaton and, therefore, is perturbative in terms of the string coupling expansion.

To proceed, we must present the structure of the heterotic string action S_z. This action was discussed in [6, 7], where we refer the reader for details and a complete description of the notation. Here, we will elucidate notation only if it is relevant to our calculation. The action is found to be

$$S_z = S_S + S_{WZW}, \tag{2.9}$$

where

$$S_S(\mathbb{Z}; \mathbb{E}_{\mathbb{M}}^{\mathbb{A}}, \mathbb{B}_{\mathbb{M}\mathbb{N}}, \phi) = T_S \int_z d\sigma^0 d\sigma^1 \left(\phi \sqrt{\det \mathbb{E}_i^A \mathbb{E}_j^B \eta_{AB}} - \frac{1}{2} \epsilon^{ij} \mathbb{E}_i^{\mathbb{A}} \mathbb{E}_j^{\mathbb{B}} \mathbb{B}_{\mathbb{B}\mathbb{A}} \right) \tag{2.10}$$

with $\mathbb{E}_i^{\mathbb{A}} \equiv (\partial_i \mathbb{Z}^{\mathbb{M}}) \mathbb{E}_{\mathbb{M}}^{\mathbb{A}}$, couples the superstring to $\mathbb{E}_{\mathbb{M}}^{\mathbb{A}}, \mathbb{B}_{\mathbb{B}\mathbb{A}}$ and ϕ, and

$$\begin{aligned}S_{WZW}(\mathbb{Z}, g; \mathbb{A}_{\mathbb{M}}) = &-\frac{1}{8\pi} \int_z d\sigma^0 d\sigma^1 \, \mathrm{tr} \left[\frac{1}{2} \sqrt{h} h^{ij} (\omega_i - \mathbb{A}_i)(\omega_j - \mathbb{A}_j) + i\epsilon^{ij} \omega_i \mathbb{A}_j \right] + \\ &+ \frac{1}{24} \int_{\mathcal{B}} d^3\hat{\sigma} i \epsilon^{\hat{i}\hat{j}\hat{k}} \, \mathrm{tr} \, \omega'_{\hat{k}} \omega'_{\hat{j}} \omega'_{\hat{i}}\end{aligned} \tag{2.11}$$

is the Wess-Zumino-Witten term coupling the superstring to the supergauge connection. Here, $\mathbb{A}_i = (\partial_i \mathbb{Z}^{\mathbb{M}})\mathbb{A}_{\mathbb{M}}$ and ω is the Lie algebra valued one-form on the heterotic string worldsheet defined by

$$\omega = g^{-1} dg \,. \tag{2.12}$$

Using (2.10) and (2.11), the calculation of the non-perturbative contribution to the two-point function (2.7) can be carried out. This calculation is technically tedious and will not be reviewed here. We refer the reader to [6, 7] for details. Suffice it to say that one proceeds by expanding each term in the action as a power series of the fermionic coordinate Θ. That is,

$$S_S = S_0 + S_\Theta + \cdots \tag{2.13}$$

and

$$S_{WZW} = S_{WZW0} + \cdots \,, \tag{2.14}$$

where we only display relevant terms. In (2.13) one finds, after some rescaling of the fields, that

$$S_0 = \frac{\mathcal{A}(z)}{2\pi\alpha'} - i \int_z B \,. \tag{2.15}$$

Here, $\mathcal{A}(z)$ is the area of the curve z measured using the pulled-back target space metric, α' is the heterotic string parameter and B is the pulled-back bosonic two-form. Furthermore, the S_Θ term can be shown to contain four-dimensional fermions λ^I that arise from the dimensional reduction on X of the ten-dimensional gravitino. After a long calculation [6, 7], one finds that

$$\begin{aligned} \langle \lambda^I(y_1)\lambda^J(y_2)\rangle \propto & \int \mathcal{D}\lambda e^{-\int \mathrm{d}^4 y \sum_K \lambda^K \partial\!\!\!/ \lambda^K} \lambda^I(y_1)\lambda^J(y_2) \\ & \int \mathrm{d}^4 x \left(\partial_L \partial_M \left(e^{-S_0} \int \mathcal{D}g e^{-S_{WZW0}}\right)\right) \lambda^L \lambda^M . \end{aligned} \tag{2.16}$$

Note that the proportionality symbol includes three determinant factors that are not relevant to the discussion in this paper. Comparing these results to the $(D_I D_J W)\lambda^I \lambda^J$ term in (2.2), and using (2.15), we conclude that the non-perturbative contribution to the superpotential of a heterotic superstring wrapped once around a holomorphic curve z is given by

$$W \propto \left(\int \mathcal{D}g e^{-S_{WZW0}}\right) e^{-\frac{\mathcal{A}(z)}{2\pi\alpha'} + i\int_z B} \,. \tag{2.17}$$

The $e^{-\frac{\mathcal{A}(z)}{2\pi\alpha'} + i\int_z B}$ factor contains only the Calabi-Yau moduli and is reasonably well-known [1, 2, 5, 6]. In this paper, we turn our attention to the factor $\int \mathcal{D}g e^{-S_{WZW0}}$, which depends on the vector bundle moduli. Although a topological criterion for the vanishing of this factor was given in [5], its functional dependence on the vector bundle moduli has only recently been computed [10].

To evaluate $\int \mathcal{D}g e^{-S_{WZW0}}$, we first note that S_{WZW0} is precisely the expression (2.11) with $\mathbb{A}_i$ replaced by the bosonic gauge connection $A_i = (\partial_i X^M)A_M$. Let us split the ten-dimensional vector field A_M into the four-and six-dimensional components A_u and A_U

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respectively. Since the holomorphic curve the superstring is wrapped on lies entirely inside the Calabi-Yau threefold X, we get

$$A_i = (\partial_i X^U) A_U \,. \tag{2.18}$$

A_U is valued in the Lie algebra of structure group $G \subseteq E_8 \times E_8$ and is a solution of the hermitean Yang-Mills equations on X. Such a solution is called a G-instanton. At this point, we note that, with some restrictions, this theory can be written in a simpler form. When $A_i = 0$, the theory described by S_{WZW0} is known [31] to be equivalent to an $\mathrm{SO}(16) \times \mathrm{SO}(16)$ theory of free Weyl fermions on z of the same chirality. Now take $A_i \neq 0$. Since only the $\mathrm{SO}(16) \times \mathrm{SO}(16)$ $(\subset E_8 \times E_8)$ symmetry is manifest in the free fermion theory, only gauge connection A_i with structure group

$$G \subseteq \mathrm{SO}(16) \times \mathrm{SO}(16) \tag{2.19}$$

can be coupled to fermions in the usual way. The coupling of gauge backgrounds with larger structure groups is far from manifest and cannot be given by a lagrangian description. For these reason we assume, henceforth, that (2.19) is satisfied. Note that, from a phenomenological point of view, this restriction is not unreasonable since, as explained in a number of papers [16], [18]–[21], realistic heterotic superstring and M-theory models always require the structure group to be contained in $\mathrm{SO}(16) \times \mathrm{SO}(16)$. With this is mind, we now write the action for $\mathrm{SO}(16) \times \mathrm{SO}(16)$ Weyl fermions coupled to a background gauge connection A_i. Denote the Weyl fermions by $\psi^a, a = 1, \ldots, 32$ where each is chosen to be a section of the negative spin bundle on z, S_-. Then

$$S_\psi = \int_z \mathrm{d}\sigma^0 \mathrm{d}\sigma^1 \, \mathrm{tr} \, \bar{\psi} D_- \psi \,, \tag{2.20}$$

where D_- is the chiral Dirac operator on z with gauge connection A. The subscript minus implies that it acts on sections of S_-. The fermion partition function is easily evaluated to give

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_\psi} \propto \det \mathcal{D}_- \,, \tag{2.21}$$

where

$$\mathcal{D}_- = \begin{pmatrix} 0 & D_- \\ i\partial_+ & 0 \end{pmatrix} \tag{2.22}$$

and $i\partial_+$ is the Dirac operator with vanishing gauge connection that acts on sections of the positive spin bundle, S_+. There is one final caveat before we can determine the partition function $\int \mathcal{D}g e^{-S_{WZW0}}$. In computing the above, we have used the S_{WZW0} action in euclidean space. Hence, the fermions to which this theory is equivalent are Weyl fermions. However, the heterotic superstring actually propagates is Minkowski space, where the associated fermions are simultaneously Weyl and Majorana spinors. These have half the number of degrees of freedom of the euclidean space Weyl fermions. It follows that, to avoid overcounting degrees of freedom, one should take the square root of the partition function. Mindful of this caveat, we see from the above discussion that

$$\int \mathcal{D}g e^{-S_{WZW0}} \propto \mathrm{Pfaff}(\mathcal{D}_-) \,, \tag{2.23}$$

where

$$\mathrm{Pfaff}(\mathcal{D}_-) = \sqrt{\det \mathcal{D}_-} \tag{2.24}$$

and $\mathcal{D}_-$ is given by (2.22). This expression is the starting point of the paper.

There are, apparently, at least two ways to evaluate $\mathrm{Pfaff}(\mathcal{D}_-)$. One approach is to view A as a gauge connection on a stable, holomorphic vector bundle V over X using the theorems of Donaldson [32] and Uhlenbeck and Yau [33]. $\mathrm{Pfaff}(\mathcal{D}_-)$ is then a global operator on z twisted by the bundle $V|_z \otimes S_-$ and $\det \mathcal{D}_-$ can be determined by purely algebraic techniques. This is the method we use in this paper. A second approach is to return to the original partition function $\int \mathcal{D}g e^{-S_{WZW_0}}$ and to evaluate it, as completely as possible, using general properties of solutions of the hermitean Yang-Mills equations. This evaluation is carried out in appendix A. One would then attempt to find explicit solutions of these equations on a Calabi-Yau threefold. These will depend, generically, on arbitrary integration constants, the vector bundle moduli. Inserting any such solution into $\int \mathcal{D}g e^{-S_{WZW_0}}$ would produce, after integration, an explicit function of the vector bundle moduli. This approach is problematic since, at present, no solutions of the hermitean Yang-Mills on a Calabi-Yau threefold are known. Be that as it may, the importance of computing the superpotentials may provide an incentive for a re-examination of this question. We emphasize that the properties of $\mathrm{Pfaff}(\mathcal{D}_-)$, such as its vanishing structure, that will be discussed in this paper might serve as a non-trivial guide to the construction of such explicit solutions.

3. The moduli of stable $\mathrm{SU}(n)$ vector bundles

Let X be an elliptically fibered Calabi-Yau threefold with base B and fiber map $\pi : X \longrightarrow B$. In this paper, we will, when specificity is required, take B to be a Hirzebruch surface

$$B = \mathbb{F}_r \,, \tag{3.1}$$

where r is any non-negative integer. As discussed in appendix B, a stable $\mathrm{SU}(n)$ holomorphic vector bundle V is determined via a Fourier-Mukai transformation from spectral data $\mathcal{C}$ and $\mathcal{N}$, where $\mathcal{C}$ is a divisor in X which is an effective, irreducible n-fold cover of the base B, and $\mathcal{N}$ is a line bundle on X. That is

$$(\mathcal{C}, \mathcal{N}) \longleftrightarrow V \,. \tag{3.2}$$

It is clear from this construction that the number of moduli of a stable $\mathrm{SU}(n)$ holomorphic vector bundle V is determined by the number of parameters specifying its spectral cover $\mathcal{C}$ and by the dimension of the space of holomorphic line bundles $\mathcal{N}$ defined on $\mathcal{C}$.

3.1 Parameters of $\mathcal{C}$ and the space of line bundles $\mathcal{N}$

The spectral cover $\mathcal{C}$ is a divisor of the Calabi-Yau threefold X and, hence, uniquely determines a line bundle $\mathcal{O}_X(\mathcal{C})$ on X. Then $H^0(X, \mathcal{O}_X(\mathcal{C}))$ is the space of holomorphic sections of $\mathcal{O}_X(\mathcal{C})$. We denote its dimension by $h^0(X, \mathcal{O}_X(\mathcal{C}))$. It follows that there must

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exist h^0 holomorphic sections $s_1, \ldots, s_{h^0}$ that span this space. Note that the zero locus of any non-vanishing element of $H^0(X, \mathcal{O}_X(\mathcal{C}))$,

$$s_{\{a_i\}} = \Sigma_{i=1}^{h^0} a_i s_i \,, \tag{3.3}$$

for fixed complex coefficients a_i, determines an effective divisor $\mathcal{C}_{\{a_i\}}$ of X in the class of $\mathcal{C}$. Running over all a_i gives the complete set, $|\mathcal{C}|$, of effective divisors in the class of $\mathcal{C}$. Clearly

$$|\mathcal{C}| = \mathbb{P}H^0(X, \mathcal{O}_X(\mathcal{C}))\,, \tag{3.4}$$

where the right hand side is the projectivization of $H^0(X, \mathcal{O}_X(\mathcal{C}))$. It follows that

$$\dim|\mathcal{C}| = h^0(X, \mathcal{O}_X(\mathcal{C})) - 1\,. \tag{3.5}$$

This quantity counts the number of parameters specifying the spectral cover $\mathcal{C}$. We now consider the line bundles $\mathcal{N}$ over $\mathcal{C}$.

The set of holomorphic line bundles $\mathcal{N}$ over the spectral cover $\mathcal{C}$ is, by definition, determined by the set of holomorphic transition functions allowed on $\mathcal{C}$. These, in turn, are specified as the closed but not exact elements of the multiplicative group $C^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}})$ of non-vanishing holomorphic functions on the intersection of any two open sets in the atlas of $\mathcal{C}$. That is, the group of line bundles of $\mathcal{C}$ is given by

$$\mathrm{Pic}(\mathcal{C}) = H^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}})\,, \tag{3.6}$$

where $H^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}})$ is the first Čech cohomology group of $\mathcal{O}^*_{\mathcal{C}}$ on $\mathcal{C}$. Clearly then, the dimension of the space of line bundles $\mathcal{N}$ over $\mathcal{C}$ is specified by

$$\dim \mathrm{Pic}(\mathcal{C}) = h^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}})\,. \tag{3.7}$$

Putting these results together, we see that the number of moduli of a stable SU(n) holomorphic vector bundle V, which we will denote by $n(V)$, is given by

$$n(V) = (h^0(X, \mathcal{O}_X(\mathcal{C})) - 1) + h^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}})\,. \tag{3.8}$$

We now turn to the explicit evaluation of each of the terms in this expression.

Our basic approach to evaluating $h^0(X, \mathcal{O}_X(\mathcal{C}))$ is through the Riemann-Roch theorem which, in this context, states that

$$\chi_E(X, \mathcal{O}_X(\mathcal{C})) = h^0(X, \mathcal{O}_X(\mathcal{C})) - h^1 + h^2 - h^3\,, \tag{3.9}$$

where $\chi_E(X, \mathcal{O}_X(\mathcal{C}))$ is the Euler characteristic and h^q for $q = 1, 2, 3$ are the dimensions of the higher cohomology groups of X evaluated in $\mathcal{O}_X(\mathcal{C})$. To proceed, we will use the fact that the stable SU(n) holomorphic vector bundles that we consider in this paper are positive. This allows us to employ the Kodaira vanishing theorem which, in turn, allows us to conclude that

$$H^q(X, \mathcal{O}_X(\mathcal{C})) = 0 \tag{3.10}$$

for $q > 0$. It follows that $h^q = 0$ for $q = 1, 2, 3$ and the Riemann-Roch theorem simplifies to

$$\chi_E(X, \mathcal{O}_X(\mathcal{C})) = h^0(X, \mathcal{O}_X(\mathcal{C})) \,. \tag{3.11}$$

Therefore, to evaluate $h^0(X, \mathcal{O}_X(\mathcal{C}))$ we need simply to evaluate the Euler characteristic. For the situation at hand, the Euler characteristic is determined from the Atiyah-Singer index theorem to be

$$\chi_E(X, \mathcal{O}_X(\mathcal{C})) = \frac{1}{6}\int_X c_1^3(\mathcal{O}_X(\mathcal{C})) + \frac{1}{12}\int_X c_1(\mathcal{O}_X(\mathcal{C})) \wedge c_2(TX) \,. \tag{3.12}$$

The first Chern class $c_1(\mathcal{O}_X(\mathcal{C}))$ is just given by $\mathcal{C}$. The second Chern class of the tangent bundle $c_2(TX)$ has been found in [34] to be

$$c_2(TX) = \pi^* \left(c_2(B) + 11c_1^2(B)\right) + 12\sigma \cdot \pi^*(c_1(B)) \,. \tag{3.13}$$

Using equation (B.8) for the first and second Chern classes of $\mathbb{F}_r$ and the spectral cover given in equations (B.9) and (B.16), one can compute $\chi_E(X, \mathcal{O}_X(\mathcal{C}))$. Of course, there are further restrictions on the integers a and b. These are 1. the non-negativity conditions (B.17) required to make $\mathcal{C}$ effective and 2. conditions (B.18), (B.19) and (B.20) necessary to render $\mathcal{C}$ irreducible and positive. With these restrictions, equation (3.11) is valid. We find that

$$h^0(X, \mathcal{O}_X(\mathcal{C})) = \frac{n}{3}(4n^2 - 1) + nab - (n^2 - 2)(a + b) + ar\left(\frac{n^2}{2} - 1\right) - \frac{n}{2}ra^2 \,. \tag{3.14}$$

We now proceed to calculate $h^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}})$.

To evaluate $h^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}})$, we note that it is an element in a long exact sequence of cohomology groups given in part by

$$\to H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \to H^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}}) \to H^2(\mathcal{C}, \mathbb{Z}) \to \,, \tag{3.15}$$

where $H^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}}) \to H^2(\mathcal{C}, \mathbb{Z})$ is the Čech coboundary operation δ_1. However, using the Lefschetz theorem one can show that

$$H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = 0 \,. \tag{3.16}$$

Hence, the mapping

$$\delta_1 : H^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}}) \to H^2(\mathcal{C}, \mathbb{Z}) \tag{3.17}$$

is injective. That is,

$$\mathrm{Pic}(\mathcal{C}) = H^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}}) \subset H^2(\mathcal{C}, \mathbb{Z}) \,. \tag{3.18}$$

Note that $H^2(\mathcal{C}, \mathbb{Z})$ forms a rigid lattice and, hence, there are no smooth deformations. We conclude that

$$\dim H^1(\mathcal{C}, \mathcal{O}^*_{\mathcal{C}}) = 0 \,. \tag{3.19}$$

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3.2 $n(V)$ for $B = \mathbb{F}_r$

We can now give the final expression for the number of moduli of a positive, stable SU(n) holomorphic vector bundle over an elliptically fibered Calabi-Yau threefold with base $B = \mathbb{F}_r$. The associated spectral cover is given by

$$\mathcal{C} = n\sigma + \pi^*(a\mathcal{S} + b\mathcal{E}) \,, \tag{3.20}$$

where the effectiveness of $\mathcal{C}$ requires

$$a \geq 0 \,, \qquad b \geq 0 \,, \tag{3.21}$$

the irreducibility of $\mathcal{C}$ demands that

$$b \geq ar \,, \qquad a \geq 2n \,, \qquad b \geq n(r+2) \tag{3.22}$$

and the positivity requirement for $\mathcal{C}$ implies

$$a > 2n \,, \qquad b > ar - n(r-2) \,. \tag{3.23}$$

Using equations (3.8), (3.14) and (3.19), one can conclude the following.

- The number of vector bundle moduli is given by

$$n(V) = \frac{n}{3}(4n^2 - 1) + nab - (n^2 - 2)(a + b) + ar\left(\frac{n^2}{2} - 1\right) - \frac{n}{2}ra^2 - 1 \,. \tag{3.24}$$

This equation will be essential to the subsequent discussion.

3.3 Vector bundles restricted to $\pi^* z$ and their moduli

In this paper, we are interested in a heterotic superstring wrapped once around a genus-zero holomorphic curve in the elliptically fibered Calabi-Yau threefold X. In fact, the curves we consider are of the form

$$\sigma \cdot \pi^* z \,, \tag{3.25}$$

where z is a holomorphic curve in the base surface B of X. We will frequently refer to $\sigma \cdot \pi^* z$ simply as z. It is clear, furthermore, that the superstring will couple, not to the complete vector bundle V but, rather, to the restriction of the vector bundle to the curve z, denoted by $V|_z$. The properties of this bundle, however, are most easily analyzed by first considering $V|_{\pi^* z}$, the restriction of V to the surface $\pi^* z$, to which we now turn.

We begin by noting that the surface $\pi^* z$ is elliptically fibered over the base curve z and that

$$\mathcal{C}|_{\pi^* z} = \mathcal{C} \cdot \pi^* z \tag{3.26}$$

is an effective, irreducible curve which is an n-fold cover of the base z. Similarly, $\mathcal{N}|_{\pi^* z}$ is the restriction of the line bundle $\mathcal{N}$ to $\pi^* z$. We will, to simplify our notation, denote these quantities by

$$C = \mathcal{C}|_{\pi^* z}, \qquad N = \mathcal{N}|_{\pi^* z} \,. \tag{3.27}$$

Given the restricted spectral data C and N, one can construct $V|_{\pi^*z}$ via a Fourier-Mukai transformation. That is,

$$(C, N) \longleftrightarrow V|_{\pi^*z} \,. \tag{3.28}$$

As we did in the previous section for V, we can ask how to define and enumerate the vector bundle moduli associated with $V|_{\pi^*z}$ (we want to count the deformations of $V|_{\pi^*z}$ which come as restrictions of deformations of V, not all abstract deformations of $V|_{\pi^*z}$). It is clear from (3.28) that the number of moduli of $V|_{\pi^*z}$ is determined by the number of parameters specifying the restricted spectral cover C and by the size of the space of restricted holomorphic line bundles N. Since the holomorphic line bundle $\mathcal{N}$ on the threefold does not have any continuous parameters, neither does its restriction. Therefore, we only need to consider the parameters of the spectral cover.

The spectral cover C is a divisor of the surface π^*z and, hence, uniquely determines a line bundle $\mathcal{O}_{\pi^*z}(C)$ on π^*z. Then $H^0(\pi^*z, \mathcal{O}_{\pi^*z}(C))$ is the space of holomorphic sections of $\mathcal{O}_{\pi^*z}(C)$. We denote its dimension by $h^0(\pi^*z, \mathcal{O}_{\pi^*z}(C))$. For the reasons discussed previously, the set, $|C|$, of all effective divisors in the class of C is given by

$$|C| = \mathbb{P}H^0(\pi^*z, \mathcal{O}_{\pi^*z}(C)) \,, \tag{3.29}$$

where the right hand side is the projectivization of $H^0(\pi^*z, \mathcal{O}_{\pi^*z}(C))$. It follows that

$$\dim |C| = h^0(\pi^*z, \mathcal{O}_{\pi^*z}(C)) - 1 \,. \tag{3.30}$$

This quantity counts the number of parameters specifying the spectral cover C. Putting the results of this subsection together, we see that the number of moduli of the restricted vector bundle $V|_{\pi^*z}$, which we will denote by $n(V|_{\pi^*z})$, is given by

$$n(V|_{\pi^*z}) = \dim |C| \tag{3.31}$$

and, hence, using (3.30) that

$$n(V|_{\pi^*z}) = h^0(\pi^*z, \mathcal{O}_{\pi^*z}(C)) - 1 \,. \tag{3.32}$$

We now turn to the explicit calculation of $h^0(\pi^*z, \mathcal{O}_{\pi^*z}(C))$.

To evaluate $h^0(\pi^*z, \mathcal{O}_{\pi^*z}(C))$ we again consider the Riemann-Roch theorem, which now becomes

$$\chi_E(\pi^*z, \mathcal{O}_{\pi^*z}(C)) = h^0(\pi^*z, \mathcal{O}_{\pi^*z}(C)) - h^1 + h^2 \,, \tag{3.33}$$

where $\chi_E(\pi^*z, \mathcal{O}_{\pi^*z}(C))$ is the Euler characteristic and $h^q, q = 1, 2$ are the dimensions of the higher cohomology groups of π^*z with coefficients in $\mathcal{O}_{\pi^*z}(C)$. As in the previous calculation of $h^0(X, \mathcal{O}_X(\mathcal{C}))$, expression (3.33) is only useful to us if h^1 and h^2 can be shown to vanish. To do this, we first need to demonstrate that $\mathcal{O}_{\pi^*z}(C)$ can be decomposed as

$$\mathcal{O}_{\pi^*z}(C) = K_{\pi^*z} \otimes L_{\pi^*z} \,, \tag{3.34}$$

where K_{π^*z} is the canonical bundle and L_{π^*z} is a positive line bundle. If this is the case, one can use the Kodaira vanishing theorem which immediately imlies that

$$h^1 = h^2 = 0 \,. \tag{3.35}$$

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Decomposition (3.34) can be shown for a wide class of curves z in $\mathbb{F}_r$. However, the calculations are complicated and unenlightening. Recall from appendix B that the effective curves $\mathcal{S}$ and $\mathcal{E}$ form a basis of $H_2(\mathbb{F}_r, \mathbb{R})$. In this paper, we will restrict our discussion to the curve $\mathcal{S}$. The extension of these results to $\mathcal{E}$ or other effective curves is straightforward, if tedious. That is, we will always choose the curve

$$z = \mathcal{S}\,, \tag{3.36}$$

when specificity is required. $\mathcal{S}$ is a holomorphic curve of genus zero and, hence, $\mathcal{S} = \mathbb{P}^1$. Furthermore, note from (B.7) that $\mathcal{S}^2 = -r$ and, therefore, $\mathcal{S}$ is an isolated holomorphic curve in $\mathbb{F}_r$ for $r = 1, 2$. It then follows that $\sigma \cdot \pi^*\mathcal{S}$ is an isolated holomorphic curve of genus zero in X. We demonstrated in [11] that

$$\pi^*\mathcal{S} = K3\,, \qquad dP_9\,, \qquad \mathbb{P}^1 \times T^2 \tag{3.37}$$

for $B = \mathbb{F}_0, \mathbb{F}_1$ and $\mathbb{F}_2$ respectively. As discussed in appendix B, for $B = \mathbb{F}_r$ the spectral cover in X is of the form

$$\mathcal{C} = n\sigma + \pi^*((a+1)\mathcal{S} + b\mathcal{E})\,, \tag{3.38}$$

where $a+1, b$ are integers satisfying the effectiveness, irreducibility and positivity conditions (B.17), (B.18), (B.19) and (B.20). Note that we use $a+1$, rather than a, as the coefficient of $\mathcal{S}$ in (3.38) to match our conventions in [11]. The coefficient $a+1$ signifies the vector bundle after the small instanton transition, when the data of a single curve $\mathcal{S}$ is added to the bundle. Using (B.7), it then follows from (3.26) and (3.27) that

$$C = n\sigma|_{\pi^*\mathcal{S}} + (b - (a+1)r)F\,. \tag{3.39}$$

We showed in [11] that

$$\mathcal{O}_{\pi^*\mathcal{S}}(C) = K_{\pi^*\mathcal{S}} \otimes L_{\pi^*\mathcal{S}}\,, \tag{3.40}$$

where

$$L_{\pi^*\mathcal{S}} = \mathcal{O}_{\pi^*\mathcal{S}}(n\sigma|_{\pi^*\mathcal{S}} + (b - ar)F)\,, \tag{3.41}$$

and that the line bundle $L_{\pi^*\mathcal{S}}$ is positive if and only if

$$b > ar - n(r-2)\,. \tag{3.42}$$

However, this is exactly the second positivity condition in (B.7). Therefore, if $\mathcal{C}$ is chosen to be positive then so is the line bundle $L_{\pi^*\mathcal{S}}$, and equation (3.35) holds. Therefore, equation (3.33) becomes

$$\chi_E(\pi^*\mathcal{S}, \mathcal{O}_{\pi^*\mathcal{S}}(C)) = h^0(\pi^*\mathcal{S}, \mathcal{O}_{\pi^*\mathcal{S}}(C))\,. \tag{3.43}$$

The Euler characteristic and, hence, $h^0(\pi^*\mathcal{S}, \mathcal{O}_{\pi^*\mathcal{S}}(C))$ can be calculated using the Atiyah-Singer index theorem. We showed in [11] that

$$h^0(\pi^*\mathcal{S}, \mathcal{O}_{\pi^*\mathcal{S}}(C)) = nb - (n^2 - 2) + \frac{rn}{2}(n-1) - r(na+1)\,. \tag{3.44}$$

As was discussed in detail in [11], $h^0(\pi^*\mathcal{S}, \mathcal{O}_{\pi^*\mathcal{S}}(C))$ has a physical interpretation as the number of "transition" moduli associated with the small instanton transition centered on the curve $z = \mathcal{S}$. As these transition moduli will not be the central focus of this paper, we refer the reader to [11] for details. What is directly relevant to this paper are the number and properties of the parameters of the projective space

$$|C| = \mathbb{P}H^0(\pi^*\mathcal{S}, \mathcal{O}_{\pi^*\mathcal{S}}(C)) \tag{3.45}$$

which specify the vector bundle moduli of $V|_{\pi^*\mathcal{S}}$. From (3.32) and (3.44) one can conclude the following.

- The number of vector bundle moduli of $V|_{\pi^*\mathcal{S}}$ is given by

$$n(V|_{\pi^*\mathcal{S}}) = nb - (n^2 - 1) + \frac{rn}{2}(n-1) - r(na+1)\,. \tag{3.46}$$

- For each choice of the line bundle $\mathcal{N}$, the vector bundle moduli of $V|_{\pi^*\mathcal{S}}$ parameterize the projective space

$$\mathbb{P}H^0(\pi^*\mathcal{S}, \mathcal{O}_{\pi^*\mathcal{S}}(C)) \simeq \mathbb{P}^{nb-(n^2-1)+\frac{rn}{2}(n-1)-r(na+1)}\,. \tag{3.47}$$

To illustrate these concepts, we now present four examples that we will explore throughout this paper. In the first two of these examples, we will choose

$$B = \mathbb{F}_1\,, \qquad G = \mathrm{SU}(3) \tag{3.48}$$

and, hence, $r = 1$, $n = 3$ and

$$\pi^*\mathcal{S} = dP_9\,. \tag{3.49}$$

It follows that (3.46) and (3.47) become

$$n(V|_{dP_9}) = 3(b-a) - 6 \tag{3.50}$$

and

$$\mathbb{P}H^0(dP_9, \mathcal{O}_{dP_9}(C)) \simeq \mathbb{P}^{3(b-a)-6} \tag{3.51}$$

respectively. Furthermore, in order to assure that $\mathcal{C}$ is positive, we will always assume that

$$a > 5\,. \tag{3.52}$$

Example 1. In this example, take

$$b - a = 5\,. \tag{3.53}$$

Then, from (3.39) we have

$$C = 3\sigma|_{dP_9} + 4F\,. \tag{3.54}$$

Note, that $a+1, b$ satisfy (B.17), (B.18) and (B.19). Therefore, C is both effective and irreducible. In addition, both conditions in (B.20) are satisfied, guaranteeing that $\mathcal{C}$ and, hence, L_{dP_9} are positive. It follows that

$$n(V|_{dP_9}) = 9 \tag{3.55}$$

and

$$\mathbb{P}H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 4F)) \simeq \mathbb{P}^9\,. \tag{3.56}$$

Example 2. In this example, choose

$$b - a = 6\,. \tag{3.57}$$

Then (3.39) implies

$$C = 3\sigma|_{dP_9} + 5F\,. \tag{3.58}$$

Integers $a+1, b$ satisfy (B.17), (B.18) and (B.19) and both conditions in (B.20). Therefore C is effective and irreducible and L_{dP_9} is positive. Then

$$n(V|_{dP_9}) = 12 \tag{3.59}$$

and

$$\mathbb{P}H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 5F)) \simeq \mathbb{P}^{12}\,. \tag{3.60}$$

In the third and the fourth examples, we will take

$$B = \mathbb{F}_2\,, \qquad G = \mathrm{SU}(3) \tag{3.61}$$

and, hence, $r = 2$, $n = 3$ and

$$\pi^*\mathcal{S} = \mathbb{P}^1 \times T^2\,. \tag{3.62}$$

It follows that (3.46) and (3.47) become

$$n(V|_{\mathbb{P}^1\times T^2}) = 3(b - 2a) - 4 \tag{3.63}$$

and

$$\mathbb{P}H^0(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(C)) \simeq \mathbb{P}^{3(b-2a)-4} \tag{3.64}$$

respectively. Furthermore, we will always take

$$a > 5\,. \tag{3.65}$$

Example 3. Choose in this case

$$b - 2a = 3\,. \tag{3.66}$$

Then, we see from (3.39) that

$$C = 3\sigma|_{\mathbb{P}^1\times T^2} + F\,. \tag{3.67}$$

Integers $a+1, b$ satisfy (B.17), (B.18) and (B.19) and both conditions in (B.20). Therefore C is effective and irreducible and $L_{\mathbb{P}^1\times T^2}$ is positive. Then

$$n(V|_{\mathbb{P}^1\times T^2}) = 5 \tag{3.68}$$

and

$$\mathbb{P}H^0(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} + F)) \simeq \mathbb{P}^5\,. \tag{3.69}$$

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Example 4. Choose in this case

$$b - 2a = 4\,. \tag{3.70}$$

Then, we see from (3.39) that

$$C = 3\sigma|_{\mathbb{P}^1\times T^2} + 2F\,. \tag{3.71}$$

Integers $a+1, b$ satisfy (B.17), (B.18) and (B.19) and both conditions in (B.20). Therefore C is effective and irreducible and $L_{\mathbb{P}^1\times T^2}$ is positive. Then

$$n(V|_{\mathbb{P}^1\times T^2}) = 8 \tag{3.72}$$

and

$$\mathbb{P}H^0(\mathbb{P}^1\times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} + 2F)) \simeq \mathbb{P}^8\,. \tag{3.73}$$

4. Vector bundles on the curve z

4.1 Vector bundle moduli on z

For the calculation of the superpotential, it is very important to know the direct image of the line bundle $\mathcal{O}_{\pi^*z}(C)$ on z, that is

$$\pi_*\mathcal{O}_{\pi^*z}(C)\,. \tag{4.1}$$

Clearly, since C is an n-fold cover of z, (4.1) will be a rank n vector bundle over z. Since, in all examples in this paper, z will be chosen to be a Riemann surface of genus zero, we can always decompose this rank n vector bundle into the direct sum of n line bundles

$$\bigoplus_{i=1}^{n}\mathcal{O}_z(m_i) \tag{4.2}$$

for some integers m_i. To proceed, we once again restrict our attention to the fundamental curve

$$z = \mathcal{S}\,. \tag{4.3}$$

The extention of these results to other effective curves z is straightforward. Then, we showed in (3.39) that C has the form

$$C = n\sigma|_{\pi^*\mathcal{S}} + fF\,, \tag{4.4}$$

where

$$f = b - (a+1)r\,. \tag{4.5}$$

It is possible to show, for

$$B = \mathbb{F}_0 \Rightarrow \pi^*\mathcal{S} = K3 \tag{4.6}$$

that

$$\pi_*\mathcal{O}_{K3}(n\sigma|_{K3} + fF) = \mathcal{O}_{\mathcal{S}}(f) \oplus \bigoplus_{i=2}^{n}\mathcal{O}_{\mathcal{S}}(f-2i)\,, \tag{4.7}$$

for

$$B = \mathbb{F}_1 \Rightarrow \pi^* \mathcal{S} = dP_9 \tag{4.8}$$

that

$$\pi_* \mathcal{O}_{dP_9}(n\sigma|_{dP_9} + fF) = \mathcal{O}_{\mathcal{S}}(f) \oplus \bigoplus_{i=2}^{n} \mathcal{O}_{\mathcal{S}}(f - i) \tag{4.9}$$

and for

$$B = \mathbb{F}_2 \Rightarrow \pi^* \mathcal{S} = \mathbb{P}^1 \times T^2 \tag{4.10}$$

that

$$\pi_* \mathcal{O}_{\mathbb{P}^1 \times T^2}(n\sigma|_{\mathbb{P}^1 \times T^2} + fF) = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathcal{S}}(f)\,. \tag{4.11}$$

The proofs of these statements are sufficiently complicated that we present them in appendix C. Expressions (4.7), (4.9) and (4.11) allow us to compute $\pi_* \mathcal{O}_{\pi^*\mathcal{S}}(C)$ explicitly as the sum of line bundles over $\mathcal{S}$. We are particularly interested in the linear space

$$H^0(\mathcal{S}\,, \pi_* \mathcal{O}_{\pi^*\mathcal{S}}(C)) \tag{4.12}$$

of holomorphic sections of the rank n vector bundle on $\mathcal{S}$ and, in particular, in its dimension

$$h^0(\mathcal{S}\,, \pi_* \mathcal{O}_{\pi^*\mathcal{S}}(C))\,. \tag{4.13}$$

- Let us first assume that $B = \mathbb{F}_0$. Then

$$h^0(K3, \pi_* \mathcal{O}_{K3}(C)) = h^0 \left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(f) \oplus \bigoplus_{i=2}^{n} \mathcal{O}_{\mathcal{S}}(f - 2i) \right). \tag{4.14}$$

  Using the fact that

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) = m + 1 \tag{4.15}$$

  for $m \geq 0$ and

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) = 0 \tag{4.16}$$

  for $m < 0$, the summation

$$\sum_{i=2}^{n} i = \frac{n^2 + n - 2}{2} \tag{4.17}$$

  and (4.5) for $r = 0$, one can easily show that

$$h^0 \left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(f) \oplus \bigoplus_{i=2}^{n} \mathcal{O}_{\mathcal{S}}(f - 2i) \right) = nb - n^2 + 2\,. \tag{4.18}$$

  Note that this is identical to expression (3.44) evaluated for $r = 0$ and, hence, $h^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(f) \oplus \bigoplus_{i=2}^{n} \mathcal{O}_{\mathcal{S}}(f - 2i))$ gives the number of transition moduli for $z = \mathcal{S}$ and $B = \mathbb{F}_0$.

- Now assume that $B = \mathbb{F}_1$. Then

$$h^0(dP_9, \pi_* \mathcal{O}_{dP_9}(C)) = h^0\left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(f) \oplus \bigoplus_{i=2}^{n} \mathcal{O}_{\mathcal{S}}(f-i)\right). \tag{4.19}$$

  Using (4.15), (4.17) and (4.5) for $r = 1$, we find that

$$h^0\left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(f) \oplus \bigoplus_{i=2}^{n} \mathcal{O}_{\mathcal{S}}(f-i)\right) = n(b-a-1) + \frac{n}{2}(1-n) + 1\,, \tag{4.20}$$

  which is identical to expression (3.44) evaluated for $r = 1$. Therefore, again, $h^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(f) \oplus \bigoplus_{i=2}^{n} \mathcal{O}_{\mathcal{S}}(f-i))$ gives the number of transition moduli for $z = \mathcal{S}$ and $B = \mathbb{F}_1$.

- Finally, consider $B = \mathbb{F}_2$. Then

$$h^0(\mathbb{P}^1 \times T^2, \pi_* \mathcal{O}_{\mathbb{P}^1 \times T^2}(C)) = h^0\left(\mathcal{S}, \bigoplus_{i=1}^{n} \mathcal{O}_{\mathcal{S}}(f)\right). \tag{4.21}$$

  Using (4.15) and (4.5) for $r = 2$, we find that

$$h^0\left(\mathcal{S}, \bigoplus_{i=1}^{n} \mathcal{O}_{\mathcal{S}}(f)\right) = n(b-2a-1)\,, \tag{4.22}$$

  which is identical to expression (3.44) evaluated for $r = 2$. Therefore, again, $h^0(\mathcal{S}, \bigoplus_{i=1}^{n} \mathcal{O}_{\mathcal{S}}(f))$ gives the number of transition moduli for $z = \mathcal{S}$ and $B = \mathbb{F}_2$.

The fact that $h^0(\mathcal{S}, \pi_* \mathcal{O}_{\pi^* \mathcal{S}}(C))$ is identical to the number of transition moduli is not coincidental. Using a Leray spectral sequence, one can show that

$$H^0(\mathcal{S}, \pi_* \mathcal{O}_{\pi^* \mathcal{S}}(C)) = H^0(\pi^* \mathcal{S}, \mathcal{O}_{\pi^* \mathcal{S}}(C)) \tag{4.23}$$

for any base $B = \mathbb{F}_r$. It follows from this that $h^0(\mathcal{S}, \pi_* \mathcal{O}_{\pi^* \mathcal{S}}(C))$ is the number of transition moduli.

To illustrate these concepts, we now consider the four examples introduced previously. In the first two examples

$$r = 1\,, \qquad n = 3\,, \qquad a > 5\,. \tag{4.24}$$

Example 1. In this example

$$b - a = 5 \tag{4.25}$$

and we see from (3.54) that

$$C = 3\sigma|_{dP_9} + 4F\,. \tag{4.26}$$

Then, equation (4.9) implies

$$\pi_* \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 4F) = \mathcal{O}_{\mathcal{S}}(4) \oplus \mathcal{O}_{\mathcal{S}}(2) \oplus \mathcal{O}_{\mathcal{S}}(1)\,. \tag{4.27}$$

Furthermore, using (4.15) we see

$$h^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(4) \oplus \mathcal{O}_{\mathcal{S}}(2) \oplus \mathcal{O}_{\mathcal{S}}(1)) = 10\,, \tag{4.28}$$

which is consistent with (3.55).

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Example 2. In this example

$$b - a = 6 \tag{4.29}$$

and from (3.58)

$$C = 3\sigma|_{dP_9} + 5F\,. \tag{4.30}$$

Then, equation (4.9) implies that

$$\pi_* \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 5F) = \mathcal{O}_{\mathcal{S}}(5) \oplus \mathcal{O}_{\mathcal{S}}(3) \oplus \mathcal{O}_{\mathcal{S}}(2)\,. \tag{4.31}$$

Furthermore, using (4.15) we see

$$h^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(5) \oplus \mathcal{O}_{\mathcal{S}}(3) \oplus \mathcal{O}_{\mathcal{S}}(2)) = 13 \tag{4.32}$$

which is consistent with (3.59).

In the third and the fourth examples, we choose

$$r = 2, \qquad n = 3, \qquad a > 5\,. \tag{4.33}$$

Example 3. In this example

$$b - 2a = 3 \tag{4.34}$$

and from (3.67)

$$C = 3\sigma|_{\mathbb{P}^1 \times T^2} + F\,. \tag{4.35}$$

Then, equation (4.11) implies that

$$\pi_* \mathcal{O}_{\mathbb{P}^1 \times T^2}(3\sigma|_{\mathbb{P}^1 \times T^2} + F) = \mathcal{O}_{\mathcal{S}}(1) \oplus \mathcal{O}_{\mathcal{S}}(1) \oplus \mathcal{O}_{\mathcal{S}}(1)\,. \tag{4.36}$$

Furthermore, using (4.15) we see

$$h^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1) \oplus \mathcal{O}_{\mathcal{S}}(1) \oplus \mathcal{O}_{\mathcal{S}}(1)) = 6\,, \tag{4.37}$$

which is consistent with (3.68).

Example 4. In this example

$$b - 2a = 4 \tag{4.38}$$

and from (3.71)

$$C = 3\sigma|_{\mathbb{P}^1 \times T^2} + 2F\,. \tag{4.39}$$

Then, equation (4.11) implies that

$$\pi_* \mathcal{O}_{\mathbb{P}^1 \times T^2}(3\sigma|_{\mathbb{P}^1 \times T^2} + 2F) = \mathcal{O}_{\mathcal{S}}(2) \oplus \mathcal{O}_{\mathcal{S}}(2) \oplus \mathcal{O}_{\mathcal{S}}(2)\,. \tag{4.40}$$

Furthermore, using (4.15) we see

$$h^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(2) \oplus \mathcal{O}_{\mathcal{S}}(2) \oplus \mathcal{O}_{\mathcal{S}}(2)) = 9\,, \tag{4.41}$$

which is consistent with (3.72).

Putting everything together, we can conclude the following.

- The number of transition moduli associated with curve z can be determined as follows. First consider the spectral cover restricted to $\pi^* z$, C. The image

$$\pi_* \mathcal{O}_{\pi^* z}(C) \tag{4.42}$$

  is a rank n vector bundle over z. The number of transition moduli associated with z is given by

$$h^0(z, \pi_* \mathcal{O}_{\pi^* z}(C)) . \tag{4.43}$$

- The number of vector bundle moduli associated with $V|_{\pi^* z}$ is given by the dimension of the projective space, that is

$$n(V|_{\pi^* z}) = h^0(z, \pi_* \mathcal{O}_{\pi^* z}(C)) - 1 . \tag{4.44}$$

4.2 The vector bundle restricted to z

As discussed previously , the superstring will couple to the restriction of the vector bundle to the curve z, $V|_z$. The number of moduli of $V|_{\pi^* z}$ was determined above and is associated on z with the image of the line bundle of C, that is, $\pi_* \mathcal{O}_{\pi^* z}(C)$. We now want to construct $V|_z$ itself. Note, however, that z is not an elliptic fibration and, hence, $V|_z$ can not be obtained by the Fourier-Mukai transformation of spectral data. To construct $V|_z$, we need to unwind what the Fourier-Mukai transformation actually means. Let $\mathcal{P}$ be the Poincare line bundle on the fiber product $X \times_B X$ and, hence, by restriction on $X \times_B \mathcal{C}$. Let π_1 and π_2 be the two projections of the fiber product $X \times_B \mathcal{C}$ onto the two factors X and $\mathcal{C}$ respectively. The fact that V is obtained by a Fourier-Mukai transformation of $(\mathcal{C}, \mathcal{N})$ means that

$$V = \pi_{1*}(\pi_2^* \mathcal{N}|_{\mathcal{C}} \otimes \mathcal{P}) . \tag{4.45}$$

Let us now consider the restriction, $V|_\sigma$, of V to the zero section $\sigma \subset X$. Clearly, using the base change theorem, this can be carried out as follows. First, restrict the line bundle $\pi_2^* \mathcal{N}|_{\mathcal{C}} \otimes \mathcal{P}$ on $X \times_B \mathcal{C}$ to

$$(\pi_2^* \mathcal{N}|_{\mathcal{C}})|_{\sigma \times_B \mathcal{C}} \otimes \mathcal{P}|_{\sigma \times_B \mathcal{C}} \tag{4.46}$$

on $\sigma \times_B \mathcal{C} \subset X \times_B \mathcal{C}$. Using the fact that

$$\sigma \times_B \mathcal{C} = \mathcal{C} , \tag{4.47}$$

this line bundle can be written simply as

$$\mathcal{N}|_{\mathcal{C}} \otimes \mathcal{P}|_{\sigma \times_B \mathcal{C}} . \tag{4.48}$$

Finally, recall that

$$\pi_{\mathcal{C}} : \mathcal{C} \to B \tag{4.49}$$

is the n-fold covering map of the spectral cover onto the base. It follows that

$$V|_B = \pi_{\mathcal{C}*}(\mathcal{N}|_{\mathcal{C}} \otimes \mathcal{P}|_{\sigma \times_B \mathcal{C}}) . \tag{4.50}$$

Identifying B with the zero section σ, and using the fact that the Poincare line bundle becomes trivial when restricted to the zero section, we see that

$$V|_{\sigma} = \pi_{\mathcal{C}*}\mathcal{N}|_{\mathcal{C}}\,. \tag{4.51}$$

We are now free to restrict this result to the curve $z \subset \sigma$. Recall that we are slightly abusing notation. What we call the curve z is, in fact, the curve $\sigma \cdot \pi^* z$ and, therefore, it is contained inside the global section σ. Using the fact that

$$\mathcal{N}|_C = N|_C \tag{4.52}$$

we obtain the following result.

- The restriction of the vector bundle to the curve z is given by

  $$V|_z = \pi_{C*}N|_C\,, \tag{4.53}$$

  where

  $$\pi_C : C \to z \tag{4.54}$$

  is the n-fold covering map of C onto the base curve z.

In all applications in this paper, the curve z on which the superstring is wrapped will be isomorphic to $\mathbb{P}^1$. It is well known that the space of line bundles over $\mathbb{P}^1$ does not have moduli since for a given first Chern class there is a unique holomorphic line bundle. On the other hand, equation (4.53) depends, by restriction, on the spectral cover, which does have continuous parameters. What happens to $V|_z$ if we start changing the parameters of C? A naive answer is that nothing happens. Since the vector bundle $V|_z$ does not have moduli, it can depend only on the homology class of the spectral cover but not on the choice of a representative inside this homology class. This answer turns out to be incorrect. In the next sections, by studying h^0 of the vector bundle $V|_z \otimes \mathcal{O}_z(-1)$, we will discover that a continuous deformation of the spectral cover inside its homology class can cause a discrete change in the topology of $V|_z$. Exploring how $h^0(z, V|_z \otimes \mathcal{O}_z(-1))$ changes as we change parameters of the spectral cover will allow us to compute the superpotential.

5. General structure of the zeros of the pfaffian

5.1 The superpotential and the pfaffian

In section 2, we showed that, in heterotic superstring theory, the superpotential is proportional to the pfaffian of the chiral Dirac operator $\mathcal{D}_-$ restricted to the curve z on which the superstring is wrapped. That is,

$$W \propto \mathrm{Pfaff}(\mathcal{D}_-)\,, \tag{5.1}$$

where

$$\mathrm{Pfaff}(\mathcal{D}_-) = \sqrt{\det \mathcal{D}_-}\,, \tag{5.2}$$

$\mathcal{D}_-$, for an appropriate choice of basis of the Clifford algebra, is given by

$$\mathcal{D}_- = \begin{pmatrix} 0 & D_- \\ i\partial_+ & 0 \end{pmatrix} \tag{5.3}$$

and

$$D_- : \Gamma(z, V|_z \otimes S_-) \to \Gamma(z, V|_z \otimes S_+)\,, \tag{5.4}$$

with

$$S = S_+ \oplus S_- \tag{5.5}$$

the spin bundle on the curve z. Since $\mathcal{D}_-$ involves $V|_z$, it follows from the above discussion that Pfaff$(\mathcal{D}_-)$ will be dependent upon the moduli of $V|_{\pi^* z}$. As discussed in [10], one can show that

$$\det \mathcal{D}_- \propto \sqrt{\det D_- D_+}\,, \tag{5.6}$$

where

$$D_+ = D_-^\dagger\,. \tag{5.7}$$

The key point in the evaluation of Pfaff$(\mathcal{D}_-)$ is to analyse under what circumstences it will vanish [5, 10]. Clearly, Pfaff$(\mathcal{D}_-)$ vanishes if and only if one or both of the operators D_- and D_+ have a non-trivial zero mode. Recall that

$$\operatorname{index} D_+ = \dim\ker D_+ - \dim\ker D_-\,. \tag{5.8}$$

From the Atiyah-Singer index theorem

$$\operatorname{index} D_+ = \frac{i}{2\pi}\int_z \operatorname{tr}\mathcal{F}\,, \tag{5.9}$$

where $\mathcal{F}$ is the curvature two-form associated with the connection $\mathcal{A}$ restricted to the curve z. Since the structure group of the holomorphic vector bundle V is contained in the semi-simple group $E_8 \times E_8$, we see that $\operatorname{tr}\mathcal{F}$ vanishes. Therefore,

$$\operatorname{index} D_+ = 0 \tag{5.10}$$

and, hence,

$$\dim\ker D_+ = \dim\ker D_-\,. \tag{5.11}$$

It follows that Pfaff$(\mathcal{D}_-)$ will vanish if and only if

$$\dim\ker D_- > 0\,. \tag{5.12}$$

By using the fact that operator D_- depends on a connection satisfying the hermitean Yang-Mills equations, one can show [10] that for a special choice of the gauge one can always set

$$D_- = i\bar{\partial}\,, \tag{5.13}$$

where, for any open neighborhood in z with coordinates σ and $\bar{\sigma}$

$$\bar{\partial} = \partial_{\bar{\sigma}}\,. \tag{5.14}$$

Therefore, Pfaff$(\mathcal{D}_-)$ will vanish if and only if

$$\dim \ker \bar{\partial} > 0 \,. \tag{5.15}$$

However, it follows from the equation (5.13) and (5.14) that the zero modes of operator $\bar{\partial}$ are precisely the holomorphic sections of the vector bundle $V|_z \otimes S_-$. Using the fact that on $z = \mathbb{P}^1$

$$S_- = \mathcal{O}_z(-1) \,, \tag{5.16}$$

and defining

$$V|_z(-1) = V|_z \otimes \mathcal{O}_z(-1) \,, \tag{5.17}$$

we see that

$$\dim \ker \bar{\partial} = h^0(z, V|_z(-1)) \,. \tag{5.18}$$

Hence, the problem of determining the zeros of the pfaffian of $\mathcal{D}_-$ is reduced to deciding whether or not there are any non-trivial global holomorphic sections of the vector bundle $V|_z(-1)$ over curve z. That is, we conclude the following.

- The pfaffian of $\mathcal{D}_-$ vanishes if and only if

$$h^0(z, V|_z(-1)) > 0 \,. \tag{5.19}$$

An equivalent way of stating the same result is to realize that the condition for the vanishing of Pfaff$(\mathcal{D}_-)$ is directly related to the non-triviality or triviality of the bundle $V|_z$. To see this, note that any holomorphic SO(16) $\times$ SO(16) vector bundle $V|_z$ over a genus zero curve $z = \mathbb{P}^1$ is of the form

$$V|_z = \bigoplus_{i=1}^{16} \mathcal{O}_{\mathbb{P}^1}(m_i) \oplus \mathcal{O}_{\mathbb{P}^1}(-m_i) \tag{5.20}$$

with non-negative integers m_i. Therefore,

$$V|_z(-1) = \bigoplus_{i=1}^{16} \mathcal{O}_{\mathbb{P}^1}(m_i - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(-m_i - 1) \,. \tag{5.21}$$

Using formulas (4.15) and (4.16) for $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_i))$ from the previous section, we see that

$$h^0(z, V|_z(-1)) = \sum_{i=1}^{16} m_i \,. \tag{5.22}$$

Therefore $h^0(z, V|_z(-1)) > 0$ if and only if at least one m_i is greater than zero. That is, $h^0(z, V|_z(-1)) > 0$ if and only if the vector bundle $V|_z$ is non-trivial.

The rest of this section will be devoted to the general discussion, in the context of stable, holomorphic vector bundles over elliptically fibered Calabi-Yau threefolds X, of when $V|_z$ is trivial and when it is non-trivial, that is, equivalently, when $V|_z(-1)$ does not have sections and when it does. The structure group of the bundle will be assumed to

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be an SU(n) subgroup of SO(16) $\times$ SO(16), although all detailed examples in this paper will, for specificity, be given for an SU(3) structure group. As we will see later in the paper, generically, the condition of triviality or non-triviality of the vector bundle $V|_z$ is not topological in the sense that it is governed not soley by the choice of the topological type of the spectral cover or the line bundle but, rather, by the choice of parameters of the spectral cover. In other words, for a given family of positive spectral covers and, therefore, a family of stable, holomorphic vector bundles on X parametrized by its moduli, the answer to the question whether or not $V|_z$ is trivial depends on the choice of the moduli. In the next section, we will give two examples of when, for every choice of the moduli, the bundle $V|_z$ is non-trivial and the superpotential vanishes identically. We will then give examples where, for a generic choice of moduli, the bundle $V|_z$ is trivial but there is a co-dimension one subspace where $V|_z$ becomes non-trivial. This means that, generically, the pfaffian and, therefore, the superpotential is non-vanishing. However, the pfaffian does vanish for moduli on the co-dimension one subspace.

5.2 Vanishing of the pfaffian

We now discuss how the choice of moduli affects the triviality or non-triviality of $V|_z$. In fact, it is technically easier to study the equivalent problem of whether or not $h^0(z, V|_z(-1))$ vanishes. In this paper, we will restrict the discussion to stable, holomorphic vector bundles over Calabi-Yau threefolds X elliptically fibered over Hirzebruch surfaces $\mathbb{F}_r$ with $r = 0, 1, 2$. The curve z will be chosen to be purely horizontal, that is, to lie in the base $B = \mathbb{F}_r$. As discussed previously, for any curve z in the base of X, the restriction, $V|_z$, of the holomorphic vector bundle to z is given by

$$V|_z = \pi_{C*}N|_C\,, \tag{5.23}$$

where C and N denote the restriction of the spectral cover $\mathcal{C}$ and line bundle $\mathcal{N}$ to the surface π^*z. Then

$$H^0(z, V|_z(-1)) = H^0(z, \pi_{C*}N|_C \otimes \mathcal{O}_z(-1)) \tag{5.24}$$

and, hence

$$h^0(z, V|_z(-1)) = h^0(z, \pi_{C*}N|_C \otimes \mathcal{O}_z(-1))\,. \tag{5.25}$$

Using a Leray spectral sequence, one can show that

$$H^0(z, \pi_{C*}N|_C \otimes \mathcal{O}_z(-1)) = H^0(C, N(-F)|_C)\,, \tag{5.26}$$

where we define

$$N(-F) = N \otimes \mathcal{O}_{\pi^*z}(-F) \tag{5.27}$$

and F is the class of the elliptic fiber. Therefore

$$h^0(z, \pi_{C*}N|_C \otimes \mathcal{O}_z(-1)) = h^0(C, N(-F)|_C)\,. \tag{5.28}$$

We conclude from (5.25) and (5.28) that one can determine when the bundle $V|_z(-1)$ has sections by studying under what circumstances the line bundle $N(-F)$, restricted to the Riemann surface C, does. To accomplish this, we consider the short exact sequence

$$0 \to E \otimes \mathcal{O}_{\pi^*z}(-D) \stackrel{f_D}{\to} E \stackrel{r}{\to} E|_D \to 0\,, \tag{5.29}$$

where E is any holomorphic vector bundle on π^*z and D is any effective divisor in π^*z. Henceforth, we will choose

$$E = N(-F) \tag{5.30}$$

and

$$D = C\,. \tag{5.31}$$

The short exact sequence (5.29) can, therefore, be written as

$$0 \to N(-F-C) \stackrel{f_C}{\to} N(-F) \stackrel{r}{\to} N(-F)|_C \to 0\,, \tag{5.32}$$

where

$$N(-F-C) = N(-F) \otimes \mathcal{O}_{\pi^*z}(-C)\,. \tag{5.33}$$

The map f_C in (5.32) is given by multiplication by a section of the line bundle $\mathcal{O}_{\pi^*z}(C)$ that vanishes precisely on C. The mapping r is just restriction. The cohomology long exact sequence corresponding to (5.32) is given by

$$\begin{aligned} 0 \to\ & H^0(\pi^*z, N(-F-C)) \to H^0(\pi^*z, N(-F)) \to H^0(C, N(-F)|_C) \to \\ & H^1(\pi^*z, N(-F-C)) \stackrel{f_C}{\to} H^1(\pi^*z, N(-F)) \to H^1(C, N(-F)|_C) \to \\ & H^2(\pi^*z, N(-F-C)) \to H^2(\pi^*z, N(-F)) \to 0\,. \end{aligned} \tag{5.34}$$

First, we note from (5.25) and (5.28) that the dimension of the cohomology group $H^0(C, N(-F)|_C)$ is precisely the quantity that we want to compute. Furthermore, since the sequence (5.34) is exact, the group $H^0(C, N(-F)|_C)$ is the kernel of the map f_C. Mapping f_C in the long exact sequence is the linear map from $H^1(\pi^*z, N(-F-C))$ to $H^1(\pi^*z, N(-F))$ induced by multiplying by a section of $\mathcal{O}_{\pi^*z}(C)$ that vanishes at C. As it arises in a similar way to f_C in the short exact sequence (5.32), we give this linear mapping the same symbol. Finally, note that the cohomology groups $H^1(\pi^*z, N(-F-C))$ and $H^1(\pi^*z, N(-F))$ are finite dimensional vector spaces and, hence, the map f_C can be represented by a matrix.

We now arrive at a key point. The restriction of the spectral cover C represents the homology class whose elements parametrize a projective space, as discussed above. As we deform a spectral cover C inside its homology class, we actually change the map f_C since we change the explicit section that vanishes at C. This implies that the map f_C depends on the vector bundle moduli. More precisely, it depends not on all vector bundle moduli but only on those that parametrize C. As we discussed in section 3, the set of parameters of C forms the projective space

$$\mathbb{P}H^0(\pi^*z, \mathcal{O}_{\pi^*z}(C))\,. \tag{5.35}$$

Therefore, the map f_C depends on the coordinates of (5.35).

Before specifying the curve z and the parameters of the line bundle $\mathcal{N}$, let us make a general comment on the cohomology exact sequence (5.34). Suppose that

$$h^0(\pi^*z, N(-F)) > h^0(\pi^*z, N(-F-C))\,. \tag{5.36}$$

Then sequence (5.34) implies that

$$h^0(C, N(-F)|_C) \tag{5.37}$$

can not be zero. Therefore, if condition (5.36) is fulfilled the line bundle $N(-F)|_C$ necessarily has sections and the pfaffian vanishes identically. We will, shortly, give an example when this situation indeed happens. However, as will be amply demonstrated, in most cases inequality (5.36) is not satisfied and the pfaffian need not be zero.

The further analysis is too complicated in the general setting, so we will make some specifications. Curve z will be chosen, as previously, to be

$$z = \mathcal{S} \tag{5.38}$$

Then, $\mathcal{S} = \mathbb{P}^1$ and, since $\mathcal{S}\cdot\mathcal{S} = -r$, it is an isolated curve in $\mathbb{F}_r$ for $r = 1, 2$. It follows that for $r = 1, 2$ the curve $\sigma \cdot \pi^*\mathcal{S}$ is $\mathbb{P}^1$ and isolated in X. Furthermore, we need to specify the line bundle $N(-F) = \mathcal{N}|_{\pi^*\mathcal{S}}(-F)$. This involves the restriction of the global line bundle $\mathcal{N}$ on X given by equation (B.21) in appendix B. That is

$$\mathcal{N} = \mathcal{O}_X\left(n\left(\frac{1}{2} + \lambda\right)\sigma + \left(\frac{1}{2} - \lambda\right)\pi^*\eta + \left(\frac{1}{2} + n\lambda\right)\pi^*c_1(B)\right), \tag{5.39}$$

where $c_1(B)$ is the first Chern class of the base $\mathbb{F}_r$

$$c_1(\mathbb{F}_r) = 2\mathcal{S} + (r+2)\mathcal{E} \tag{5.40}$$

and η is the curve on the base specifying the spectral cover. Following the notation of the previous sections, we write

$$\eta = (a+1)\mathcal{S} + b\mathcal{E}\,. \tag{5.41}$$

The coefficients a and b are chosen in such a way that the spectral cover

$$\mathcal{C} = n\sigma + \pi^*\eta \tag{5.42}$$

satisfies effectiveness, irreducibility and positivity conditions (B.17)–(B.20). The restriction of the spectral cover to $\pi^*\mathcal{S}$ was given in (3.39) to be

$$C = n\sigma|_{\pi^*\mathcal{S}} + (b - (a+1)r)F\,. \tag{5.43}$$

Furthermore, the restriction of the line bundle $\mathcal{N}$ to $\pi^*\mathcal{S}$ is of the form

$$N = \mathcal{N}|_{\pi^*\mathcal{S}} = \mathcal{O}_{\pi^*\mathcal{S}}\left(\left(n\left(\frac{1}{2} + \lambda\right)\sigma + \left(\frac{1}{2} - \lambda\right)\pi^*\eta + \left(\frac{1}{2} + n\lambda\right)\pi^*c_1(B)\right)\cdot\pi^*\mathcal{S}\right). \tag{5.44}$$

Using equations (5.44), (5.40), (5.41) and the intersecton properties (B.7) in appendix B, we obtain that

$$N(-F) = \mathcal{O}_{\pi^*\mathcal{S}}\left(n\left(\lambda + \frac{1}{2}\right)\sigma|_{\pi^*\mathcal{S}} + \left(\left(\frac{1}{2} - \lambda\right)(b - (a+1)r) + \left(\frac{1}{2} + n\lambda\right)(2-r) - 1\right)F\right). \tag{5.45}$$

We begin our analysis by giving an explicit example of when Pfaff$(\mathcal{D}_-)$ and, hence, the superpotential vanishes identically.

- Suppose that the following conditions on the parameters are fulfilled

$$\left(\frac{1}{2}-\lambda\right)(b-(a+1)r)+\left(\frac{1}{2}+n\lambda\right)(2-r)-1=0\,,\qquad \lambda+\frac{1}{2}\geq 1\,. \tag{5.46}$$

  Then it follows that the line bundles $N(-F)$ and $N(-F-C)$ are of the form

$$N(-F)=\mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}})\,,\qquad A>0 \tag{5.47}$$

  and

$$N(-F-C)=\mathcal{O}_{\pi^*\mathcal{S}}(A'\sigma|_{\pi^*\mathcal{S}}-B'F)\,,\qquad A'\geq 0, B'>0 \tag{5.48}$$

  respectively. The condition $B'>0$ is the consequence of the fact that we work with positive spectral covers, which implies $b-ar>0$. Using a Leray spectral sequence, it follows that

$$H^0(\pi^*\mathcal{S},\mathcal{O}_{\pi^*\mathcal{S}}(A'\sigma|_{\pi^*\mathcal{S}}-B'F))=H^0(\mathcal{S},\pi_*\mathcal{O}_{\pi^*\mathcal{S}}(A'\sigma|_{\pi^*\mathcal{S}}-B'F))\,. \tag{5.49}$$

  Furthermore, equation (C.16) in appendix C implies

$$H^0(\pi^*\mathcal{S},\mathcal{O}_{\pi^*\mathcal{S}}(A'\sigma|_{\pi^*\mathcal{S}}-B'F))=H^0\left(\mathcal{S},\mathcal{O}_{\mathcal{S}}(-B')\oplus\bigoplus_{i=2}^{A'}(-B'+i(r-2))\right)\,. \tag{5.50}$$

  Since $B'>0$, we see from equation (4.16) that the line bundle $\mathcal{O}_{\pi^*\mathcal{S}}(A'\sigma|_{\pi^*\mathcal{S}}-B'F)$ does not have any global holomorphic sections. On the other hand, the same analysis, using equation (C.15) in appendix C, shows that the line bundle $\mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}})$ has a unique section for $r=0,1$ and A sections for $r=2$. Therefore, we have proven that conditions (5.36) are fulfilled whenever equations (5.46) are satisfied. This, in turn, implies that if equations (5.46) are satisfied, the pfaffian and, therefore, the superpotential induced by a superstring wrapped on a holomorphic curve $\mathcal{S}$ vanish identically.

Parameters satisfying (5.46), as well as the effectiveness, irreducibility and positivity conditions (B.17), (B.18), (B.19) and (B.20), are easily found. An explicit example is

$$r=1\,,\qquad n=3\,,\qquad b-a=4\,,\qquad \lambda=\frac{3}{2}\,. \tag{5.51}$$

We now turn to the more interesting situation when there is no obvious reason for the pfaffian to vanish identically. In fact, several different cases have to be distinguished. They are

$$\lambda>\frac{1}{2}\,,\qquad \lambda<-\frac{1}{2}\,,\qquad \lambda=\frac{1}{2}\,,\qquad \lambda=-\frac{1}{2}\,. \tag{5.52}$$

- Let us first consider the case

$$\lambda > \frac{1}{2}\,. \tag{5.53}$$

Here, of course, we assume that the first condition in (5.46) is not fulfilled since otherwise, as we have just shown, the pfaffian would vanish identically. If condition (5.53) is satisfied, one can show, using Leray spectral sequences and equations (5.43) and (5.45), that the second cohomology groups in (5.34) can be related to the second cohomology groups on the curve $\mathcal{S}$ and, hence, vanish identically. That is

$$H^2(\pi^*\mathcal{S}, N(-F-C)) = H^2(\mathcal{S}, \pi_* N(-F-C)) = 0 \tag{5.54}$$

and

$$H^2(\pi^*\mathcal{S}, N(-F)) = H^2(\mathcal{S}, \pi_* N(-F)) = 0\,. \tag{5.55}$$

Here, we have used the fact that

$$R^1\pi_*\mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}} + fF) = 0 \tag{5.56}$$

for $A > 0$ and any f. To simplify the sequence (5.34), let us assume that the line bundle $N(-F)$ will always be of the form

$$\mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}} - BF)\,, \qquad A > 0\,, \quad B > 0\,. \tag{5.57}$$

This condition will be satisfied in all the examples we will consider. As explained above, such line bundles have no global holomorphic sections and, hence

$$H^0(\pi^*\mathcal{S}, N(-F)) = 0\,. \tag{5.58}$$

Then, the general structure of the exact sequence (5.34) simplifies to

$$\begin{aligned} 0 \to H^0(C, N(-F)|_C) \to H^1(\pi^*\mathcal{S}, N(-F-C)) \overset{f_C}{\to} \\ H^1(\pi^*\mathcal{S}, N(-F)) \to H^1(C, N(-F)|_C) \to 0\,. \end{aligned} \tag{5.59}$$

Now, let us note the following fact. From equations (5.24)-(5.28) we know that

$$H^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) = H^0(C, N(-F)|_C) \tag{5.60}$$

and therefore, that

$$h^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) = h^0(C, N(-F)|_C)\,. \tag{5.61}$$

On the other hand, using a Leray spectral sequence, it follows that the first cohomology groups satisfy the same relation, that is

$$H^1(\mathcal{S}, V|_{\mathcal{S}}(-1)) = H^1(C, N(-F)|_C) \tag{5.62}$$

and, hence

$$h^1(\mathcal{S}, V|_{\mathcal{S}}(-1)) = h^1(C, N(-F)|_C)\,. \tag{5.63}$$

The quantity

$$h^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) - h^1(\mathcal{S}, V|_{\mathcal{S}}(-1))\,, \tag{5.64}$$

as follows from the Atiyah-Singer theorem and the fact that $\mathcal{O}_{\mathcal{S}}(-1)$ is a spin bundle, is nothing but the index of the Dirac operator $\mathcal{D}_-$ that has been shown previously to vanish. This allows us to conclude that

$$h^0(C, N(-F)|_C) = h^1(C, N(-F)|_C)\,. \tag{5.65}$$

It then follows from sequence (5.59) that the middle cohomology groups must have the same dimension. That is,

$$h^1(\pi^*\mathcal{S}, N(-F-C)) = h^1(\pi^*\mathcal{S}, N(-F))\,. \tag{5.66}$$

This says that f_C in (5.59) is a mapping between two linear spaces of the same dimension and, therefore, can be represented by a square matrix. A non-trivial kernel of the matrix f_C is equivalent to the condition of the determinant of f_C vanish. We conclude, therefore, that

$$h^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) > 0 \tag{5.67}$$

if and only if

$$\det f_C = 0\,. \tag{5.68}$$

As we have previously discussed, the matrix f_C depends on the vector bundle moduli associated with the restriction of the spectral cover $\mathcal{C}$ to the surface $\pi^*\mathcal{S}$. Therefore, $\det f_C$ is a function of these moduli. Within the zero locus of $\det f_C$

$$h^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) > 0\,, \tag{5.69}$$

the vector bundle $V|_{\mathcal{S}}$ is non-trivial and the pfaffian vanishes. However, outside this zero locus

$$h^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) = 0 \tag{5.70}$$

and, as a result, the vector bundle $V|_{\mathcal{S}}$ is trivial and the pfaffian does not vanish.

To illustrate these concepts, we now consider the examples introduced in the previous section. Since we have restricted λ to satisfy (5.53), we need only consider examples 2, 3 and 4 above. We will continue to refer to them with the same numbering. The $\det f_C$ in these examples will be explicity computed in the next section.

Example 2. In this example, we take

$$r = 1\,, \qquad n = 3\,, \qquad a > 5 \tag{5.71}$$

and, hence, the lift of the curve $\mathcal{S}$ is the rational elliptic surface dP_9. The bundle over it is specified by

$$b - a = 6\,, \qquad \lambda = \frac{3}{2}\,. \tag{5.72}$$

Note that λ satisfies (5.53). Then, from (4.30)

$$C = 3\sigma|_{dP_9} + 5F\,. \tag{5.73}$$

Furthermore, it follows from (5.45) that the line bundle $N(-F)$ is given by

$$N(-F) = \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F)\,. \tag{5.74}$$

The line bundle $N(-F)$ is of the form (5.57) and, therefore,

$$H^0(dP_9, N(-F)) = 0\,. \tag{5.75}$$

Since condition (5.53) is satisfied, the second cohomology groups in the exact sequence (5.34) vanish. Therefore, sequence (5.34) is reduced to (5.59) which, using equations (5.60), (5.62), (5.73) and (5.74), can be written as

$$0 \to H^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to W_1 \stackrel{f_C}{\to} W_2 \to H^1(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to 0\,, \tag{5.76}$$

where

$$W_1 = H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F)) \tag{5.77}$$

and

$$W_2 = H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))\,. \tag{5.78}$$

The construction of the map f_C, which will be given in the next section, is based on the fact that one can relate cohomology groups on dP_9 to cohomology groups on $\mathcal{S} = \mathbb{P}^1$. This is achieved by a Leray spectral sequence which implies that

$$H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F)) = H^1(\mathcal{S}, \pi_*\mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F)) \tag{5.79}$$

and

$$H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F)) = H^1(\mathcal{S}, \pi_*\mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))\,. \tag{5.80}$$

From equations (4.9), (5.77) and (5.79), it follows that

$$W_1 \simeq H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-6) \oplus \mathcal{O}_{\mathcal{S}}(-8) \oplus \mathcal{O}_{\mathcal{S}}(-9)) \tag{5.81}$$

and

$$W_2 \simeq H^1\left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-1) \oplus \bigoplus_{i=3}^{7} \mathcal{O}_{\mathcal{S}}(-i)\right)\,. \tag{5.82}$$

Furthermore, by using the Serre duality

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2-p))^* \tag{5.83}$$

we can rewrite W_1 and W_2 as

$$W_1 \simeq H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(4) \oplus \mathcal{O}_{\mathcal{S}}(6) \oplus \mathcal{O}_{\mathcal{S}}(7))^* \tag{5.84}$$

and

$$W_2 \simeq H^0\left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-1) \oplus \bigoplus_{i=3}^{7} \mathcal{O}_{\mathcal{S}}(-2+i)\right)^*\,. \tag{5.85}$$

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From equations (4.15) and (4.16), we conclude that

$$\dim W_1 = \dim W_2 = 20\,. \tag{5.86}$$

Therefore, the linear map f_C in this example can be represented by a 20×20 matrix. This matrix depends on the parameters of the spectral cover restricted to dP_9 given in (5.73). From equations (3.60), (4.31) and (4.32), we know that the space of these parameters is the projective space

$$\mathbb{P}H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 5F)) \simeq \mathbb{P}H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(5) \oplus \mathcal{O}_{\mathcal{S}}(3) \oplus \mathcal{O}_{\mathcal{S}}(2)) \simeq \mathbb{P}^{12}\,. \tag{5.87}$$

This means that the map f_C is parametrized by the coordinates of the twelve-dimensional projective space (5.87).

Example 3. In this example, we take

$$r = 2\,, \qquad n = 3\,, \qquad a > 5 \tag{5.88}$$

and, hence, the lift of the curve $\mathcal{S}$ is just the cross product $\mathbb{P}^1 \times T^2$. The vector bundle on this surface is specified by

$$b - 2a = 3\,, \qquad \lambda = \frac{3}{2}\,. \tag{5.89}$$

Note that λ satisfies condition (5.53). Then, from equation (4.35) it follows that

$$C = 3\sigma|_{\mathbb{P}^1 \times T^2} + F\,. \tag{5.90}$$

Furthermore, it follows from (5.45) that the line bundle $N(-F)$ is given by

$$N(-F) = \mathcal{O}_{\mathbb{P}^1 \times T^2}(6\sigma|_{\mathbb{P}^1 \times T^2} - 2F)\,. \tag{5.91}$$

As in the previous example,

$$H^0(\mathbb{P}^1 \times T^2, N(-F)) = 0 \tag{5.92}$$

and the second cohomology groups vanish

$$H^2(\mathbb{P}^1 \times T^2, N(-F - C)) = H^2(\mathbb{P}^1 \times T^2, N(-F)) = 0\,. \tag{5.93}$$

Thus, the corresponding cohomology exact sequence becomes

$$0 \to H^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to W_3 \xrightarrow{f_C} W_4 \to H^1(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to 0\,, \tag{5.94}$$

where

$$W_3 = H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1 \times T^2}(3\sigma|_{\mathbb{P}^1 \times T^2} - 3F)) \tag{5.95}$$

and

$$W_4 = H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1 \times T^2}(6\sigma|_{\mathbb{P}^1 \times T^2} - 2F))\,. \tag{5.96}$$

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Using (4.11) and Serre duality (5.83), these cohomology groups can be related to those on the curve $\mathcal{S}$ as

$$W_3 \simeq H^1\left(\mathcal{S}, \bigoplus_{i=1}^{3} \mathcal{O}_{\mathcal{S}}(-3)\right) \simeq H^0\left(\mathcal{S}, \bigoplus_{1}^{3} \mathcal{O}_{\mathcal{S}}(1)\right)^* \tag{5.97}$$

and

$$W_4 \simeq H^1\left(\mathcal{S}, \bigoplus_{i=1}^{6} \mathcal{O}_{\mathcal{S}}(-2)\right) \simeq H^0\left(\mathcal{S}, \bigoplus_{i=1}^{6} \mathcal{O}_{\mathcal{S}}\right)^* . \tag{5.98}$$

It follows from (4.15) and (4.16) that

$$\dim W_3 = \dim W_4 = 6 . \tag{5.99}$$

Therefore, the linear map f_C in this example can be represented by a 6×6 matrix. The restriction of the spectral cover is of the form (5.90) and, as follows from equations (3.69), (4.36) and (4.37), its moduli parametrize the projective space

$$\mathbb{P}H^0(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} + F)) \simeq \mathbb{P}H^0\left(\mathcal{S}, \bigoplus_{i=1}^{3} \mathcal{O}_{\mathcal{S}}(1)\right) \simeq \mathbb{P}^5 . \tag{5.100}$$

Hence, the map f_C depends on six homogeneous coordinates.

Example 4. In this example, we take

$$r = 2 , \qquad n = 3 , \qquad a > 5 \tag{5.101}$$

and, hence, the lift of the curve $\mathcal{S}$ is just the cross product $\mathbb{P}^1 \times T^2$. The vector bundle on this surface is specified by

$$b - 2a = 4 , \qquad \lambda = \frac{3}{2} . \tag{5.102}$$

Note that λ satisfies condition (5.53). Then, from (4.39)

$$C = 3\sigma|_{\mathbb{P}^1\times T^2} + 2F . \tag{5.103}$$

Furthermore, it follows from (5.45) that the line bundle $N(-F)$ is given by

$$N(-F) = \mathcal{O}_{\mathbb{P}^1\times T^2}(6\sigma|_{\mathbb{P}^1\times T^2} - 3F) . \tag{5.104}$$

As in the previous example,

$$H^0(\mathbb{P}^1 \times T^2, N(-F)) = 0 \tag{5.105}$$

and the second cohomology groups vanish

$$H^2(\mathbb{P}^1 \times T^2, N(-F-C)) = H^2(\mathbb{P}^1 \times T^2, N(-F)) = 0 . \tag{5.106}$$

Thus, the corresponding cohomology exact sequence becomes

$$0 \to H^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to W_5 \overset{f_C}{\to} W_6 \to H^1(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to 0 , \tag{5.107}$$

where

$$W_5 = H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} - 5F)) \tag{5.108}$$

and

$$W_6 = H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(6\sigma|_{\mathbb{P}^1\times T^2} - 3F))\,. \tag{5.109}$$

Using (4.11) and the Serre duality (5.83), these cohomology groups can be related to those on the curve $\mathcal{S}$ as

$$W_5 \simeq H^1\left(\mathcal{S}, \bigoplus_{i=1}^{3} \mathcal{O}_{\mathcal{S}}(-5)\right) \simeq H^0\left(\mathcal{S}, \bigoplus_{1}^{3} \mathcal{O}_{\mathcal{S}}(3)\right)^* \tag{5.110}$$

and

$$W_6 \simeq H^1\left(\mathcal{S}, \bigoplus_{i=1}^{6} \mathcal{O}_{\mathcal{S}}(-3)\right) \simeq H^0\left(\mathcal{S}, \bigoplus_{i=1}^{6} \mathcal{O}_{\mathcal{S}}(1)\right)^*. \tag{5.111}$$

It follows from (4.15) and (4.16) that

$$\dim W_5 = \dim W_6 = 12\,. \tag{5.112}$$

Therefore, the linear map f_C in this example can be represented by a 12×12 matrix. The restriction of the spectral cover is of the form (5.103) and, as follows from equations (3.73), (4.40) and (4.41), its moduli parametrize the projective space

$$\mathbb{P}H^0(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} + 2F)) \simeq \mathbb{P}H^0\left(\mathcal{S}, \bigoplus_{i=1}^{3} \mathcal{O}_{\mathcal{S}}(2)\right) \simeq \mathbb{P}^8\,. \tag{5.113}$$

Hence, the map f_C depends on nine homogeneous coordinates.

- We now turn to the second case in (5.52), where

$$\lambda < -\frac{1}{2}\,. \tag{5.114}$$

In this case, the procedure presented above has to be slightly modified. The reason is that the coefficient in front of $\sigma|_{\pi^*\mathcal{S}}$ in (5.45) is now negative. We showed in (5.65) that

$$h^0(C, N(-F)|_C) = h^1(C, N(-F)|_C)\,. \tag{5.115}$$

On the other hand, by Serre duality

$$h^1(C, N(-F)|_C) = h^0(C, (N(-F))^*|_C \otimes K_C)\,, \tag{5.116}$$

where by K_C we denote the canonical bundle of C. Thanks to equations (5.115) and (5.116), instead of studying the question of when

$$h^0(C, N(-F)|_C) > 0\,, \tag{5.117}$$

we can study the equivalent question of when

$$h^0(C, (N(-F))^*|_C \otimes K_C) > 0\,. \tag{5.118}$$

This observation is helpful for the following reason. By the adjunction formula, the canonical bundle K_C can be related to the canonical bundle $K_{\pi^*\mathcal{S}}$ by

$$K_C = (K_{\pi^*\mathcal{S}} \otimes \mathcal{O}_{\pi^*\mathcal{S}}(C))|_C\,. \tag{5.119}$$

The canonical bundle of $\pi^*\mathcal{S}$ was shown in [11] to be

$$K_{\pi^*\mathcal{S}} = \mathcal{O}_{\pi^*\mathcal{S}}(-rF)\,. \tag{5.120}$$

Then, using (5.43), one finds that

$$K_C = \mathcal{O}_{\pi^*\mathcal{S}}(n\sigma|_{\pi^*\mathcal{S}} + (b-(a+2)r)F)|_C\,. \tag{5.121}$$

If condition (5.114) is satisfied, it is easy to see that

$$(N(-F))^* \otimes K_{\pi^*\mathcal{S}} \otimes \mathcal{O}_{\pi^*\mathcal{S}}(C) = \mathcal{O}_{\pi^*\mathcal{S}}(A''\sigma|_{\pi^*\mathcal{S}} + \cdots)\,, \tag{5.122}$$

where

$$A'' = n\left(-\lambda + \frac{1}{2}\right) > 0 \tag{5.123}$$

and the ellipses stand for the fiber class term, which is irrelevant to our argument. The fact that A'' is positive allows us to proceed with our discussion. Return to the short exact sequence (5.29). Although we continue to choose $D = C$, the vector bundle E is now taken to be

$$E = (N(-F))^* \otimes K_{\pi^*\mathcal{S}} \otimes \mathcal{O}_{\pi^*\mathcal{S}}(C)\,. \tag{5.124}$$

Note that

$$E|_C = (N(-F))^*|_C \otimes K_C\,. \tag{5.125}$$

The procedure discussed above is now repeated using this different data.

Repeating this procedure in the general setting requires that we introduce rather complicated notation. Here, therefore, it is simplest to work within the context of an explicit example. Let us consider example 1 discussed previously.

Example 1. In this example, we take

$$r = 1\,, \qquad n = 3\,, \qquad a > 5 \tag{5.126}$$

and, hence, the lift of the curve $\mathcal{S}$ is just dP_9. The vector bundle on this surface is specified by

$$b - a = 5\,, \qquad \lambda = -\frac{5}{2}\,. \tag{5.127}$$

Note that λ satisfies (5.114). Then, from (4.26)

$$C = 3\sigma|_{dP_9} + 4F\,. \tag{5.128}$$

Furthermore, it follows from (5.45) that the line bundle $N(-F)$ is given by

$$N(-F) = \mathcal{O}_{dP_9}(-6\sigma|_{dP_9} + 4F) \tag{5.129}$$

The coefficient in front of $\sigma|_{dP_9}$ is negative, so we introduce the "Serre dual" version of $N(-F)|_C$

$$(N(-F))^*|_C \otimes K_C\,. \tag{5.130}$$

In the case under consideration

$$K_C = \mathcal{O}_{dP_9}(3\sigma|_{\pi^*\mathcal{S}} + 3F)|_C \tag{5.131}$$

and, therefore,

$$(N(-F))^*|_C \otimes K_C = \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)|_C\,. \tag{5.132}$$

We see then that an equivalent condition for the vanishing of the pfaffian is given by

$$h^0(C, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)|_C) > 0\,. \tag{5.133}$$

Since the coefficient in front of $\sigma|_{dP_9}$ in (5.133) is positive, we can now apply the techniques developed previously in this section. The new expression for E given in (5.124) becomes, in this example,

$$E = \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)\,. \tag{5.134}$$

Then, the exact sequence (5.29) is given by

$$0 \to \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - 5F) \stackrel{f_C}{\to} \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F) \to \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)|_C \to 0\,. \tag{5.135}$$

The cohomology long exact sequence associated with (5.135) is of the form

$$\begin{aligned} 0 \to\ & H^0(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - 5F)) \to H^0(dP_9, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)) \to \\ & H^0(C, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)|_C) \to H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - 5F)) \stackrel{f_C}{\to} \\ & H^1(dP_9, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)) \to H^1(C, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)|_C) \to \\ & H^2(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - 5F)) \to H^2(dP_9, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)) \to 0\,. \end{aligned} \tag{5.136}$$

The line bundle $\mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)$ is of the type (5.57) and, therefore, does not have sections. That is

$$H^0(dP_9, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)) = 0\,. \tag{5.137}$$

By using a Leray spectral sequence, it is easy to see that second cohomology groups in (5.136) vanish

$$H^2(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - 5F)) = H^2(dP_9, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)) = 0\,. \tag{5.138}$$

As a result, sequence (5.136) is reduced to

$$0 \to H^0(C, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)|_C) \to W_7 \stackrel{f_C}{\to} W_8 \to H^1(C, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)|_C) \to 0\,, \tag{5.139}$$

where

$$W_7 = H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - 5F)) \tag{5.140}$$

and

$$W_8 = H^1(dP_9, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F))\,. \tag{5.141}$$

Using equation (4.9) and a Leray spectral sequence, we find that

$$W_7 \simeq H^1\left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-5) \oplus \bigoplus_{i=7}^{11} \mathcal{O}_{\mathcal{S}}(-i)\right) \tag{5.142}$$

and

$$W_8 \simeq H^1\left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-1) \oplus \bigoplus_{i=3}^{10} \mathcal{O}_{\mathcal{S}}(-i)\right). \tag{5.143}$$

By Serre duality (5.83), we can also rewrite W_7 and W_8 in the form

$$W_7 \simeq H^0\left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(3) \oplus \bigoplus_{i=5}^{9} \mathcal{O}_{\mathcal{S}}(i)\right)^* \tag{5.144}$$

and

$$W_8 \simeq H^0\left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-1) \oplus \bigoplus_{i=1}^{8} \mathcal{O}_{\mathcal{S}}(i)\right)^*. \tag{5.145}$$

It then follows from equations (4.15), (4.16) that

$$\dim W_7 = \dim W_8 = 44\,. \tag{5.146}$$

Therefore, the linear map f_C in this example can be represented by a 44×44 matrix. It follows from equations (3.56), (4.27) and (4.28) that the restriction of the spectral cover to dP_9 given in (5.128) has moduli which parameterize the projective space

$$\mathbb{P}H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 4F)) \simeq \mathbb{P}H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(4) \oplus \mathcal{O}_{\mathcal{S}}(2) \oplus \mathcal{O}_{\mathcal{S}}(1)) \simeq \mathbb{P}^9\,. \tag{5.147}$$

Hence, the matrix f_C is parametrized by the coordinates of the nine-dimensional projective space (5.147).

The remaining two cases in (5.52),

$$\lambda = \frac{1}{2} \tag{5.148}$$

and

$$\lambda = -\frac{1}{2} \tag{5.149}$$

require a substantial modification of the procedure discussed above and will not be considered in this paper.

6. Computing $\det f_C$

In this section, for the four examples involving isolated curve $\mathcal{S}$ presented in the previous sections, we will show how to explicitly find the matrix f_C whose kernel is $h^0(\mathcal{S}, V|_{\mathcal{S}}(-1))$. We then compute the determinant $\det f_C$ and give it a global interpretation. This will allow us to obtain an explicit expression for the superpotential. The preliminary set-up for constructing f_C was given in the previous section.

Let us begin with

Example 2. In this example, we take

$$r = 1\,, \qquad n = 3\,, \qquad a > 5 \tag{6.1}$$

and, hence, $\pi^*\mathcal{S} = dP_9$. The bundle over this surface is specified by

$$b - a = 6\,, \qquad \lambda = \frac{3}{2}\,. \tag{6.2}$$

Let us recall the basic set-up from the previous section. The sequence that we are going to study was given in (5.76) by

$$0 \to H^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to W_1 \overset{f_C}{\to} W_2 \to H^1(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to 0\,, \tag{6.3}$$

where

$$W_1 = H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F)) \tag{6.4}$$

and

$$W_2 = H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))\,. \tag{6.5}$$

The dimension of each of the linear spaces linear spaces W_1 and W_2 is 20 and, hence, the map f_C can be represented by a 20×20 matrix. This matrix depends on the parameters of the spectral cover restricted to dP_9, that is, on the homogeneous coordinates of

$$\mathbb{P}H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 5F)) \simeq \mathbb{P}H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(5) \oplus \mathcal{O}_{\mathcal{S}}(3) \oplus \mathcal{O}_{\mathcal{S}}(2)) \simeq \mathbb{P}^{12}\,. \tag{6.6}$$

We will extensively use the relation of W_1 and W_2 to the cohomology groups on the curve $\mathcal{S}$ (equations (5.81), (5.82), (5.84) and (5.85)), given by

$$W_1 \simeq H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-6) \oplus \mathcal{O}_{\mathcal{S}}(-8) \oplus \mathcal{O}_{\mathcal{S}}(-9)) \simeq H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(4) \oplus \mathcal{O}_{\mathcal{S}}(6) \oplus \mathcal{O}_{\mathcal{S}}(7))^* \tag{6.7}$$

and

$$W_2 \simeq H^1\left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-1) \oplus \bigoplus_{i=3}^{7} \mathcal{O}_{\mathcal{S}}(-i)\right) \simeq H^0\left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-1) \oplus \bigoplus_{i=3}^{7} \mathcal{O}_{\mathcal{S}}(-2+i)\right)^*\,. \tag{6.8}$$

The main idea in constructing the matrix f_C is to express all data on dP_9 in terms of data on the curve $\mathcal{S}$ using the isomorphisms (6.7) and (6.8). Let us denote by

$$\tilde{w}_1 = B_{-6} \oplus B_{-8} \oplus B_{-9} \tag{6.9}$$

an arbitrary element of $H^1(\mathcal{S},\mathcal{O}_{\mathcal{S}}(-6)\oplus\mathcal{O}_{\mathcal{S}}(-8)\oplus\mathcal{O}_{\mathcal{S}}(-9))$, where $B_{-i}, i=6,8,9$, represents an arbitrary differential in $H^1(\mathcal{S},\mathcal{O}_{\mathcal{S}}(-i))$. We see from the Serre duality relation (5.83) and (4.15) that

$$h^1(\mathcal{S},\mathcal{O}_{\mathcal{S}}(-i)) = i-1\,. \tag{6.10}$$

Let us now lift $\tilde{w}_1$ to an element w_1 in $H^1(dP_9,\mathcal{O}_{dP_9}(3\sigma|_{dP_9}-6F))$ and denote by $b_i=\pi^*B_i$ the lift of B_i. Clearly, w_1 is given by a sum of b_i multiplied by global sections s_1,s_2,s_3 of some line bundles on dP_9. That is

$$w_1 = b_{-6}s_1 + b_{-8}s_2 + b_{-9}s_3\,. \tag{6.11}$$

It is easy to see that

$$\begin{aligned} &s_1 \in H^0(dP_9,\mathcal{O}_{dP_9}(3\sigma|_{dP_9}))\,,\\ &s_2 \in H^0(dP_9,\mathcal{O}_{dP_9}(3\sigma|_{dP_9}+2F))\,,\\ &s_3 \in H^0(dP_9,\mathcal{O}_{dP_9}(3\sigma|_{dP_9}+3F))\,. \end{aligned} \tag{6.12}$$

Note that the line bundle $\mathcal{O}_{dP_9}(3\sigma|_{dP_9})$ has only one section vanishing along $\sigma|_{dP_9}$ with multiplicity 3. This section is identical [11, 34] to z in the Weierstrass equation for dP_9

$$y^2z = 4x^3 - g_2xz^2 - g_3z^3\,. \tag{6.13}$$

That is,

$$s_1 = z\,. \tag{6.14}$$

Furthermore, it was shown in [11] that

$$x \in H^0(dP_9,\mathcal{O}_{dP_9}(3\sigma|_{dP_9}+2F))\,, \qquad y \in H^0(dP_9,\mathcal{O}_{dP_9}(3\sigma|_{dP_9}+3F)) \tag{6.15}$$

and

$$g_2 \in H^0(dP_9,\mathcal{O}_{dP_9}(4F))\,, \qquad g_3 \in H^0(dP_9,\mathcal{O}_{dP_9}(6F))\,. \tag{6.16}$$

We see, therefore, that the sections s_2 and s_3 can be taken to be

$$s_2 = x\,, \qquad s_3 = y\,. \tag{6.17}$$

It follows that

$$w_1 = b_{-6}z + b_{-8}x + b_{-9}y\,. \tag{6.18}$$

Expression (6.18) completely characterizes an element $w_1 \in W_1$.

In a similar way, any element $w_2 \in W_2 = H^1(dP_9,\mathcal{O}_{dP_9}(6\sigma|_{dP_9}-F))$ can be written as

$$w_2 = c_{-3}zx + c_{-4}zy + c_{-5}x^2 + c_{-6}xy + c_{-7}y^2 \tag{6.19}$$

where for $j=3,\ldots,7$, $c_{-j}=\pi^*C_{-j}$ is an element of $H^1(dP_9,\mathcal{O}_{dP_9}(-jF))$ and C_{-j} is a section in the $j-1$-dimensional space $H^1(\mathcal{S},\mathcal{O}_{\mathcal{S}}(-j))$. The map f_C is can also be expressed in terms of data on curve $\mathcal{S}$. By using (6.6) and (6.17), f_C can be written as

$$f_C = m_5z + m_3x + m_2y\,, \tag{6.20}$$

where $m_k = \pi^* M_k$, $k = 2, 3, 5$, is an element in $H^0(dP_9, \mathcal{O}_{dP_9}(kF))$ and M_k is a section in the $k+1$-dimensional space $H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(k))$. Although there are thirteen parameters in m_k, $k = 2, 3, 5$, it must be remembered that they are homogeneous coordinates for the twelve dimensional projective space $\mathbb{P}H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 5F))$.

Putting this all together, we can completely specify the linear mapping $W_1 \xrightarrow{f_C} W_2$. First, note that with respect to fixed basis vectors of W_1 and W_2, the linear map f_C is a 20×20 matrix. In order to find this matrix explicitly, we have to study its action on these basis vectors. This action is generated through multiplcation by a section f_C of the form (6.20). Suppressing, for the time being, the vector coefficients b_{-i} and c_{-j}, we see from (6.11) that the linear space W_1 is spanned by the basis vector blocks

$$z\,, \qquad x\,, \qquad y \tag{6.21}$$

whereas it follows from (6.19) that the linear space W_2 is spanned by basis vector blocks

$$zx\,, \qquad zy\,, \qquad x^2\,, \qquad xy\,, \qquad y^2\,. \tag{6.22}$$

The explicit matrix M_{IJ} representing f_C is determined by multiplying the basis vectors (6.21) of W_1 by f_C in (6.20). Expanding the resulting vectors in W_2 in the basis (6.22) yields the matrix. We find that M_{IJ} is given by

$$\begin{array}{c c} & \begin{array}{ccc} z & x & y \end{array} \\ \begin{array}{c} xz \\ yz \\ x^2 \\ xy \\ y^2 \end{array} & \begin{pmatrix} m_3 & m_5 & 0 \\ m_2 & 0 & m_5 \\ 0 & m_3 & 0 \\ 0 & m_2 & m_3 \\ 0 & 0 & m_2 \end{pmatrix} \end{array}. \tag{6.23}$$

Of course, M_{IJ} is a 20×20 matrix, so each of the elements of (6.23) represents a $(j-1) \times (i-1)$ matrix for the corresponding $j = 3, 4, 5, 6, 7$ and $i = 6, 8, 9$. Note, that if we multiply $m_5 z$ by a basis vector block z we get $m_5 z^2$ which is not in the space W_2. Therefore, it does not represent a holomorphic differential and should be put to zero. For example, let us compute M_{11}. This corresponds to the $xz - z$ component of (6.23) where

$$H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F))|_{b_{-6}} \xrightarrow{m_3} H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))|_{c_{-3}}\,. \tag{6.24}$$

Note, from (6.10), that

$$h^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F))|_{b_{-6}} = 5 \tag{6.25}$$

and

$$h^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))|_{c_{-3}} = 2\,. \tag{6.26}$$

An explicit matrix for m_3 is most easily obtained if we use equations (6.4), (6.7) and (6.5), (6.8) to identify

$$H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F))|_{b_{-6}} = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(4))^* \tag{6.27}$$

and

$$H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))|_{c_{-3}} = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1))^* . \tag{6.28}$$

If we define the two dimensional linear space

$$\hat{V} = H^0(\mathcal{S}, \mathcal{O}(1)) , \tag{6.29}$$

then we see that

$$H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F))|_{b_{-6}} = \mathrm{Sym}^4\, \hat{V}^* \tag{6.30}$$

and

$$H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))|_{c_{-3}} = \hat{V}^* , \tag{6.31}$$

where by $\mathrm{Sym}^k\, \hat{V}^*$ we denote the k-th symmetrized tensor product of the dual vector space $\hat{V}^*$ of $\hat{V}$. Similarly, it follows from (6.6) and (6.29) that m_3 is an element in

$$H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 5F))|_{m_3} = \mathrm{Sym}^3\, \hat{V} . \tag{6.32}$$

Let us now introduce a basis

$$\{u, v\} \in \hat{V} \tag{6.33}$$

and the dual basis

$$\{u^*, v^*\} \in \hat{V}^* , \tag{6.34}$$

where

$$u^*u = v^*v = 1 , \qquad u^*v = v^*u = 0 . \tag{6.35}$$

Then the space $\mathrm{Sym}^k\, \hat{V}^*$ is spanned by all possible homogeneous polynomials in u^*, v^* of degree k. Specifically,

$$\{u^{*4}, u^{*3}v^*, u^{*2}v^{*2}, u^*v^{*3}, v^{*4}\} \in \mathrm{Sym}^4\, \hat{V}^* \tag{6.36}$$

is a basis of $\mathrm{Sym}^4\, \hat{V}^*$ and

$$\{u^3, u^2v, uv^2, v^3\} \in \mathrm{Sym}^3\, \hat{V} . \tag{6.37}$$

is a basis of $\mathrm{Sym}^3\, \hat{V}$. Clearly, any section m_3 can be written in the basis (6.37) as

$$m_3 = \phi_1 u^3 + \phi_2 u^2 v + \phi_3 u v^2 + \phi_4 v^3 , \tag{6.38}$$

where $\phi_a, a = 1, \cdots 4$ represent the associated moduli. Now, by using the multiplication rules (6.35), we find that the explicit 2×5 matrix representation of m_3 in this basis, that is, the M_{11} submatrix of M_{IJ}, is given by

$$\begin{array}{c} \\ u^* \\ v^* \end{array} \begin{array}{c} \begin{array}{ccccc} u^{*4} & u^{*3}v^* & u^{*2}v^{*2} & u^*v^{*3} & v^{*4} \end{array} \\ \left(\begin{array}{ccccc} \phi_1 & \phi_2 & \phi_3 & \phi_4 & 0 \\ 0 & \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{array} \right) \end{array} . \tag{6.39}$$

Continuing in this manner, we can fill out the complete 20×20 matrix M_{IJ}. It is not particularly enlightening, so we will not present the matrix M_{IJ} in this paper. What is important is the determinant of M_{IJ}. Let us parametrize the sections m_2 and m_5 as

$$\begin{aligned} m_2 &= \chi_1 u^2 + \chi_2 uv + \chi_3 v^2 \\ m_5 &= \psi_1 u^5 + \psi_2 u^4 v + \psi_3 u^3 v^2 + \psi_4 u^2 v^3 + \psi_5 u v^4 + \psi_6 v^5 \end{aligned} \tag{6.40}$$

where χ_b, $b = 1,2,3$ and ψ_c, $c = 1,\dots 6$ represent the associated moduli. It is then straightforward to compute the determinant of M_{IJ} using the programs Maple or MATHEMATICA. We find that it has two rather remarkable properties. First, it factors in a simple way. And second, it is independent of some of the variables. Specifically, we find the following.

- The determinant of f_C is given by

$$\det f_C = \det M_{IJ} = \mathcal{P}^4\,, \tag{6.41}$$

where

$$\begin{aligned}\mathcal{P} = {} & \chi_1^2\chi_3\phi_3^2 - \chi_1^2\chi_2\phi_3\phi_4 - 2\chi_1\chi_3^2\phi_3\phi_1 - \\ & - \chi_1\chi_2\chi_3\phi_3\phi_2 + \chi_2^2\chi_3\phi_1\phi_3 + \phi_4^2\chi_1^3 - \\ & - 2\phi_2\phi_4\chi_3\chi_1^2 + \chi_1\chi_3^2\phi_2^2 + 3\phi_1\phi_4\chi_1\chi_2\chi_3 + \\ & + \phi_2\chi_1\phi_4\chi_2^2 + \phi_1^2\chi_3^3 - \phi_2\chi_2\phi_1\chi_3^2 - \phi_4\phi_1\chi_2^3\end{aligned} \tag{6.42}$$

is a homogeneous polynomial of degree 5 in the seven projective parameters ϕ_a and χ_b. Note that none of the remaining six moduli ψ_a appear in $\mathcal{P}$.

We conclude this example by pointing out that $\det f_C$ will vanish precisely on the zero locus of the homogeneous polynomial $\mathcal{P}$. Globally, $\mathcal{P}$ must be a holomorphic section of some complex line bundle over $\mathbb{P}^{12}$. Therefore, there must exist a divisor $D_{\mathcal{P}} \subset \mathbb{P}^{12}$ such that $\mathcal{P}$ is a section of

$$\mathcal{O}_{\mathbb{P}^{12}}(D_{\mathcal{P}}) \tag{6.43}$$

and vanishes on the co-dimension one submanifold $D_{\mathcal{P}}$ of $\mathbb{P}^{12}$. This categorizes the space of zeroes of $\mathcal{P}$ and, hence, $\det f_C$. The exact eleven dimensional submanifold

$$D_{\mathcal{P}} \subset \mathbb{P}H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(5) \oplus \mathcal{O}_{\mathcal{S}}(3) \oplus \mathcal{O}_{\mathcal{S}}(2)) \tag{6.44}$$

can be determined by solving the equation $\mathcal{P} = 0$ using (6.42).

As a second example, let us consider

Example 1. In this example, we take

$$r = 1\,, \qquad n = 3\,, \qquad a > 5 \tag{6.45}$$

and, hence, $\pi^*\mathcal{S} = dP_9$. The vector bundle over dP_9 is specified by

$$b - a = 5\,, \qquad \lambda = -\frac{5}{2}\,. \tag{6.46}$$

According to the previous section, the cohomology exact sequence that we need to study is

$$0 \to H^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to W_7 \stackrel{f_C}{\to} W_8 \to H^1(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to 0\,, \tag{6.47}$$

where

$$W_7 = H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - 5F)) \tag{6.48}$$

and

$$W_8 = H^1(dP_9, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F)) \,. \tag{6.49}$$

The relation of W_7 and W_8 to the cohomology groups on the curve $\mathcal{S}$ is the following

$$W_7 \simeq H^1\left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-5) \oplus \bigoplus_{i=7}^{11} \mathcal{O}_{\mathcal{S}}(-1)\right) \simeq H^0\left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(3) \oplus \bigoplus_{i=5}^{9} \mathcal{O}_{\mathcal{S}}(i)\right)^* \tag{6.50}$$

and

$$W_8 \simeq H^1\left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-1) \oplus \bigoplus_{i=3}^{10} \mathcal{O}_{\mathcal{S}}(-i)\right) \simeq H^0\left(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-1) \oplus \bigoplus_{i=1}^{8} \mathcal{O}_{\mathcal{S}}(i)\right)^* \,. \tag{6.51}$$

The dimension of each of the linear spaces W_7 and W_8 is 44 and, hence, the map f_C can be represented by a 44×44 matrix. This matrix depends on the parameters of the spectral cover restricted to dP_9, that is, on the homogeneous coordinates of

$$\mathbb{P}H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 4F)) \simeq \mathbb{P}H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(4) \oplus \mathcal{O}_{\mathcal{S}}(2)\mathcal{O}_{\mathcal{S}}(1)) \simeq \mathbb{P}^9 \,. \tag{6.52}$$

An arbitrary element $\tilde{w}_7 \in H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-5) \oplus \bigoplus_{i=7}^{11} \mathcal{O}_{\mathcal{S}}(-1))$ can be written as

$$\tilde{w}_7 = B_{-5} \oplus \bigoplus_{i=7}^{11} B_{-i} \,, \tag{6.53}$$

where $B_{-i}, i = 5, 7, \ldots, 11$ denote an arbitrary element of the $i-1$-dimensional space $H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-i))$. Let $b_i = \pi^* B_i$ represent the lift of B_i. We can now write an element $w_7 \in H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - 5F))$ as a sum of b_i multiplied by global sections of appropriate line bundles on dP_9. These global sections can be chosen to be $z^2, xz, yz, x^2, xy, x^3 z^{-1}$. We find that

$$w_7 = b_{-5} z^2 + b_{-7} xz + b_{-8} yz + b_{-9} x^2 + b_{-10} xy + b_{-11} x^3 z^{-1} \,. \tag{6.54}$$

Note that a section $x^3 z^{-1}$ is holomorphic since x vanishes along $\sigma|_{dP_9}$ with multiplicity one [34]. This cancels the pole coming from z^{-1}. Using (6.15), it is easy to see that that every term in (6.54) is an element in $H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - 5F))$. In a similar way, any element $w_8 \in W_8$ can be expressed as

$$w_8 = c_{-3} x z^2 + c_{-4} y z^2 + c_{-5} x^2 z + c_{-6} xyz + c_{-7} x^3 + c_{-8} x^2 y + c_{-9} x^4 z^{-1} c_{-10} x^3 y z^{-1} \,, \tag{6.55}$$

where for $j = 3, \ldots 10$, $c_{-j} = \pi^* C_{-j}$ is an element of $H^1(dP_9, \mathcal{O}_{dP_9}(-jF))$ and C_{-j} is a section in the $j-1$-dimensional space $H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-j))$. We now need to express the map f_C in terms of data on the curve $\mathcal{S}$. By using (6.52), f_C can be written as

$$f_C = m_4 z + m_2 x + m_1 y \,, \tag{6.56}$$

where $m_k = \pi^* M_k$, $k = 1, 2, 4$, is an element of $H^0(dP_9, \mathcal{O}_{dP_9}(kF))$ and M_k is a section in the $k+1$-dimensional space $H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(k))$. Although there are ten parameters in m_k, $k = 1, 2, 4$, they are homogeneous coordinates for the nine dimensional projective space $\mathbb{P}H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 4F))$.

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Putting this all together, we can completely specify the linear mapping $W_7 \overset{f_C}{\rightarrow} W_8$. First, note that with respect to fixed basis vectors of W_7 and W_8, linear map f_C is a 44×44 matrix. To find this matrix explicitly, we have to study its action on these basis vectors. This action is generated through multiplication by a section f_C of the form (6.56). Suppressing, for the time being, the vector coefficients b_{-i} and c_{-i}, we see from (6.54) that the linear space W_7 is spanned by the following basis vector blocks

$$z^2\,, \qquad xz\,, \qquad yz\,, \qquad x^2\,, \qquad xy\,, \qquad x^3z^{-1} \tag{6.57}$$

whereas it follows from (6.55) that the linear space W_8 is spanned by basis vector blocks

$$xz^2\,, \qquad yz^2\,, \qquad x^2z\,, \qquad xyz\,, \qquad x^3\,, \qquad x^2y\,, \qquad x^4z^{-1}\,, \qquad x^3yz^{-1}\,. \tag{6.58}$$

The explicit matrix M_{IJ} representing f_C is obtained by multiplying the basis vectors (6.57) by f_C in (6.56). Expanding the resulting vectors in W_8 in the basis (6.58) yields the matrix. We find that M_{IJ} is given by

$$\begin{array}{c c}
 & \begin{array}{cccccc} z^2 & xz & yz & x^2 & xy & x^3z^{-1} \end{array} \\
\begin{array}{c} xz^2 \\ yz^2 \\ x^2z \\ xyz \\ x^3 \\ x^2y \\ x^4z^{-1} \\ x^3yz^{-1} \end{array} &
\begin{pmatrix}
m_2 & m_4 & -m_1g_2 & 0 & -m_1g_3 & 0 \\
m_1 & 0 & m_4 & 0 & 0 & 0 \\
0 & m_2 & 0 & m_4 & -m_1g_2 & 0 \\
0 & m_1 & m_2 & 0 & m_4 & 0 \\
0 & 0 & 4m_1 & m_2 & 0 & m_4 \\
0 & 0 & 0 & m_1 & m_2 & 0 \\
0 & 0 & 0 & 0 & 4m_1 & m_2 \\
0 & 0 & 0 & 0 & 0 & m_1
\end{pmatrix}
\end{array}. \tag{6.59}$$

This matrix has a more complicated structure than the one in the previous example and requires some explanation. First, note that if we multiply z^2 by m_4z we obtain m_4z^3, which is not in the space W_8. This implies that we must set this element to zero. Second, if we multiply yz by m_1y we get m_1y^2z, which is also not among the basis vector blocks in (6.58). On the other hand, from the Weierstass equation (6.13), it follows that

$$m_1zy^2 = 4m_1x^3 - m_1g_2xz^2 - m_1g_3z^3\,. \tag{6.60}$$

Therefore, we get non-vanishing coefficients in the basis vector blocks x^3, xz^2 and z^3. This is the reason for the appearance of the entries $4m_1, -m_1g_2, -m_1g_3$ in (6.59). Third, the matrix (6.59) depends not only on the vector bundle moduli but also on the moduli of the underlying Calabi-Yau space contained in g_2 and g_3. Now, M_{IJ} is a 44×44 matrix so every entry in (6.59) represents a matrix block of dimension $(j-1) \times (i-1)$ for the corresponding $j = 3, \ldots, 10$ and $i = 5, 7, \ldots, 11$. We will not write the whole matrix but, rather, show how to explicitly compute two blocks, one with and one without Calabi-Yau moduli. For example, let us compute M_{11}. This corresponds to the $xz^2 - z^2$ component of (6.59) where

$$H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - 5F))|_{b_{-5}} \overset{m_2}{\rightarrow} H^1(dP_9, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F))|_{c_{-3}}\,. \tag{6.61}$$

Note from (6.10) that

$$h^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))|_{b_{-5}} = 4 \tag{6.62}$$

and

$$h^1(dP_9, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F))|_{c_{-3}} = 2\,. \tag{6.63}$$

We now relate the data on dP_9 to data on curve $\mathcal{S}$. Using (6.48)-(6.51), we can identify

$$H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))|_{b_{-5}} = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(3))^* \tag{6.64}$$

and

$$H^1(dP_9, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F))|_{c_{-3}} = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1))^*\,. \tag{6.65}$$

In terms of the two dimensional linear space (6.29), we see that

$$H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - 5F))|_{b_{-5}} = \mathrm{Sym}^3\,\hat{V}^* \tag{6.66}$$

and

$$H^1(dP_9, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F))|_{c_{-3}} = \hat{V}^*\,. \tag{6.67}$$

Similarly, it follows from (6.29) and (6.52) that m_2 is an element in

$$H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 4F))|_{m_2} = \mathrm{Sym}^3\,\hat{V}\,. \tag{6.68}$$

Now introduce the basis' defined in (6.33), (6.34) and (6.35). Then

$$\{u^{*3}, u^{*2}v^*, u^*v^{*2}, v^{*3}\} \in \mathrm{Sym}^3\,\hat{V}^* \tag{6.69}$$

is a basis for $\mathrm{Sym}^3\,\hat{V}^*$ and

$$\{u^2, uv, v^2\} \in \mathrm{Sym}^2\,\hat{V}\,. \tag{6.70}$$

is a basis for $\mathrm{Sym}^2\,\hat{V}$. Furthermore, any section m_2 can be written as

$$m_2 = \lambda_1 u^2 + \lambda_2 uv + \lambda_3 v^2\,, \tag{6.71}$$

where $\lambda_a, a = 1, 2, 3$, represent the associated moduli. Using the multiplication rules (6.35), we find that the explicit 2×4 matrix representation of m_2 in this basis, that is, the M_{11} submatrix of M_{IJ}, is given by

$$\begin{array}{c} \\ u^* \\ v^* \end{array} \begin{array}{c} \begin{array}{cccc} u^{*3} & u^{*2}v^* & uv^{*2} & v^{*2} \end{array} \\ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} \end{array}. \tag{6.72}$$

Let us now compute a matrix block containing Calabi-Yau moduli, for example, M_{31}. This corresponds to the $xz^2 - yz$ component of (6.59), where

$$H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - 5F))|_{b_{-8}} \stackrel{-m_1 g_2}{\to} H^1(dP_9, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F))|_{c_{-3}}\,. \tag{6.73}$$

Note from (6.10) that

$$h^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - 5F))|_{b_{-8}} = 7 \tag{6.74}$$

and

$$h^1(dP_9, \mathcal{O}_{dP_9}(9\sigma|_{dP_9} - F))|_{c_{-3}} = 2\,. \tag{6.75}$$

Using equations (6.48) and (6.50), we can identify

$$H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - 5F))|_{b_{-8}} = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(6))^* = \mathrm{Sym}^6\,\hat{V}^*\,. \tag{6.76}$$

The basis for $\mathrm{Sym}^6\,V^*$ is given by

$$\{u^{*6}, u^{*5}v^*, u^{4*}v^{*2}, u^{*3}v^{*3}, u^{*2}v^{*4}, u^*v^{*5}, v^{*6}\} \in \mathrm{Sym}^6\,\hat{V}^*\,. \tag{6.77}$$

A section m_1 is an element in

$$H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 4F))|_{m_1} = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1)) = \hat{V} \tag{6.78}$$

and can be written as

$$m_1 = \rho_1 u + \rho_2 v\,, \tag{6.79}$$

where $\rho_b, b = 1, 2$, are the associated moduli. Recall from (6.16) that g_2 is a section of $\mathcal{O}_{dP_9}(4F)$. Since

$$H^0(dP_9, \mathcal{O}_{dP_9}(4F)) \simeq H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(4)) \tag{6.80}$$

we have

$$g_2 = h_1u^4 + h_2u^3v + h_3u^2v^2 + h_4uv^3 + h_5v^4\,, \tag{6.81}$$

where $h_d, d = 1, \ldots, 5$ are the associated Calabi-Yau moduli. Using the multiplication rules (6.35), we find that the 2×7 matrix representation of m_1g_2 is given by

$$\begin{array}{c} \\ u^* \\ v^* \end{array} \begin{array}{c} \begin{array}{ccccccc} u^{*6} & u^{*5}v^* & u^{*4}v^{*2} & u^{*3}v^{*3} & u^{*2}v^{*4} & u^*v^{*5} & v^{*6} \end{array} \\ \begin{pmatrix} \rho_1h_1 & \rho_1h_2 + \rho_2h_1 & \rho_1h_3 + \rho_2h_2 & \rho_1h_4 + \rho_2h_3 & \rho_1h_5 + \rho_2h_4 & \rho_2h_5 & 0 \\ 0 & \rho_1h_1 & \rho_1h_2 + \rho_2h_1 & \rho_1h_3 + \rho_2h_2 & \rho_1h_4 + \rho_2h_3 & \rho_1h_5 + \rho_2h_4 & \rho_2h_5 \end{pmatrix} \end{array}. \tag{6.82}$$

Continuing in this manner, we can fill out the complete 44×44 matrix M_{IJ}. Let us parametrize the remaining sections m_4 as

$$m_4 = \mu_1u^4 + \mu_2u^3v + \mu_3u^2v^2 + \mu_4uv^3 + \mu_5v^4\,, \tag{6.83}$$

where $\mu_c, c = 1, \ldots, 5$ represent the associated moduli. Also parametrize

$$g_3 = t_1u^6 + t_2u^5v + t_3u^4v^2 + t_4u^3v^3 + t_5u^2v^4 + t_6uv^5 + t_7v^6\,, \tag{6.84}$$

where $t_e, e = 1, \ldots, 7$ are the associated Calabi-Yau moduli. It is then straightforward to compute the determinant of M_{IJ} using the programs Maple or MATHEMATICA. The result is the following.

- The determinant of f_C is

$$\det f_C = \det M_{IJ} = 2^{20}\mathcal{Q}^{11}\tilde{\mathcal{Q}}\,, \tag{6.85}$$

where

$$\mathcal{Q} = \rho_1 \rho_2 \lambda_2 - \lambda_1 \rho_2^2 - \lambda_3 \rho_1^2 \tag{6.86}$$

and

$$\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}(\lambda_a, \rho_b, \mu_c, h_d) \tag{6.87}$$

is a polynomial of degree 11 depending on all the vector bundle moduli λ_a, ρ_b, μ_c and on the Calabi-Yau moduli h_d but is independent of the Calabi-Yau moduli t_e. Polynomial $\mathcal{Q}$ has a very complicated form and, hence, we present it in appendix D.

Let us emphasize the main differences between this example and the previous one.

- In this example $\det f_C$ contains two factors, one of them appearing with multiplicity one. This is a very important difference since, later on, we will argue that the vector bundle moduli superpotential precisely is equal to $\det f_C$.
- Unlike in the first example, the zero locus of $\det f_C$ depends not just on vector bundle moduli but also on some of the moduli of the underlying Calabi-Yau threefold.

We conclude this example by pointing out that $\det f_C$ will vanish precisely on the union of the zero loci of the homogeneous polynomials $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$. Globally, $\mathcal{Q}\tilde{\mathcal{Q}}$ must be a holomorphic section of some complex line bundle over $\mathbb{P}^9$. Therefore, there must exist a divisor $D_{\mathcal{Q}\tilde{\mathcal{Q}}} \subset \mathbb{P}^9$ such that $\mathcal{Q}\tilde{\mathcal{Q}}$ is a section of

$$\mathcal{O}_{\mathbb{P}^9}(D_{\mathcal{Q}\tilde{\mathcal{Q}}}) \tag{6.88}$$

and vanishes on the co-dimension one submanifold $D_{\mathcal{Q}\tilde{\mathcal{Q}}}$ of $\mathbb{P}^9$. This categorizes the space of zeroes of the product $\mathcal{Q}\tilde{\mathcal{Q}}$ and, hence, $\det f_C$. The exact eight-dimensional submanifold

$$D_{\mathcal{Q}\tilde{\mathcal{Q}}} \subset \mathbb{P}H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(4) \oplus \mathcal{O}_{\mathcal{S}}(2) \oplus \mathcal{O}_{\mathcal{S}}(1)) \tag{6.89}$$

can be determined by solving the equation $\mathcal{Q}\tilde{\mathcal{Q}} = 0$ using (6.86) and (D.1) from appendix D.

As a third example, consider

Example 3. Here we take

$$r = 2\,, \qquad n = 3\,, \qquad a > 5 \tag{6.90}$$

and, hence, $\pi^*\mathcal{S} = \mathbb{P}^1 \times T^2$. The vector bundle on this surface is specified by

$$b - 2a = 3\,, \qquad \lambda = \frac{3}{2}\,. \tag{6.91}$$

Recall from the previous section, that the cohomology exact sequence under consideration is

$$0 \to H^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to W_3 \stackrel{f_C}{\to} W_4 \to H^1(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to 0\,, \tag{6.92}$$

where

$$W_3 = H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1 \times T^2}(3\sigma|_{\mathbb{P}^1 \times T^2} - 3F)) \tag{6.93}$$

and

$$W_4 = H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(6\sigma|_{\mathbb{P}^1\times T^2} - 2F)) \,. \tag{6.94}$$

The relation of W_3 and W_4 to the cohomology groups on $\mathcal{S}$ is the following

$$W_3 \simeq H^1\left(\mathcal{S}, \bigoplus_{i=1}^{3} \mathcal{O}_{\mathcal{S}}(-3)\right) \simeq H^0\left(\mathcal{S}, \bigoplus_{1}^{3} \mathcal{O}_{\mathcal{S}}(1)\right)^* \tag{6.95}$$

and

$$W_4 \simeq H^1\left(\mathcal{S}, \bigoplus_{i=1}^{6} \mathcal{O}_{\mathcal{S}}(-2)\right) \simeq H^0\left(\mathcal{S}, \bigoplus_{i=1}^{6} \mathcal{O}_{\mathcal{S}}\right)^* \,. \tag{6.96}$$

The dimension of each of the linear spaces W_3 and W_4 is 6 and, hence, the linear map f_C can be represented by a 6×6 matrix. This matrix depends on the parameters of the spectral cover restricted to $\mathbb{P}^1 \times T^2$, that is, on the six homogeneous coordinates of the projective space

$$\mathbb{P}H^0(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} + F)) \simeq \mathbb{P}H^0\left(\mathcal{S}, \bigoplus_{i=1}^{3} \mathcal{O}_{\mathcal{S}}(1)\right) \simeq \mathbb{P}^5 \,. \tag{6.97}$$

An arbitrary element $\tilde{w}_3 \in H^1(\mathcal{S}, \bigoplus_{i=1}^3 \mathcal{O}_{\mathcal{S}}(-3))$ can be written as

$$\tilde{w}_3 = B^{(1)}_{-3} \oplus B^{(2)}_{-3} \oplus B^{(3)}_{-3} \,, \tag{6.98}$$

where $B^{(i)}_{-3}, i = 1, 2, 3$, are elements of the two-dimensional vector space $H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-3))$. Let $b^{(i)}_{-3} = \pi^* B^{(i)}_{-3}$ be the lift of $B^{(i)}_{-3}$. We can now write any element $w_3 \in H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} - 3F))$ as

$$w_3 = b^{(1)}_{-3} z + b^{(2)}_{-3} x + b^{(3)}_{-3} y \,. \tag{6.99}$$

In a similar way, any element $w_4 \in H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(6\sigma|_{\mathbb{P}^1\times T^2} - 2F))$ can be expressed as

$$w_4 = c^{(1)}_{-2} z^2 + c^{(2)}_{-2} xz + c^{(3)}_{-2} yz + c^{(4)}_{-2} x^2 + c^{(5)}_{-2} xy + c^{(6)}_{-2} y^2 \,, \tag{6.100}$$

where for $j = 1, \ldots, 6$, $c^{(j)}_{-2} = \pi^* C^{(j)}_{-2}$ are elements of the one-dimensional vector space $H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(-2F))$ and $C^{(j)}_{-2}$ are elements of $H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-2))$. We now need to express the map f_C in terms of the data on the curve $\mathcal{S}$. By using (6.97), f_C can be written as

$$f_C = m_1 z + m_2 x + m_3 y \,, \tag{6.101}$$

where $m_k = \pi^* M_k, k = 1, 2, 3$, are elements of $H^0(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(F))$ and M_k are elements of the two-dimensional vector space $H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1))$. Although there are six parameters m_k, they are homogeneous coordinates for the five-dimensional projective space $\mathbb{P}H^0(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} + F))$.

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Putting this all together, we can specify the linear mapping $W_3 \overset{f_C}{\to} W_4$. With respect to fixed basis vectors of W_3 and W_4, the linear map f_C is a 6×6 matrix. To find this matrix, we have to study its action on these basis vectors. The action is generated through multiplication by a section f_C of the form (6.101). Suppressing, for the time being, the vector coefficients $b^{(i)}_{-3}$ and $c^{(i)}_{-2}$, we see from (6.99) that the linear space W_3 is spanned by the following basis vector blocks

$$z\,, \qquad x\,, \qquad y \tag{6.102}$$

whereas it follows from (6.100) that the linear space W_4 is spanned by basis vector blocks

$$z^2\,, \qquad xz\,, \qquad yz\,, \qquad x^2\,, \qquad xy\,, \qquad y^2\,. \tag{6.103}$$

The explicit matrix M_{IJ} is obtained by multiplying the basis vector blocks (6.102) by f_C in (6.101). Expanding the resulting vectors in W_4 in the basis (6.103) yields the matrix. We find that M_{IJ} is given by

$$\begin{array}{c} \\ z^2 \\ xz \\ yz \\ x^2 \\ xy \\ y^2 \end{array} \begin{array}{c} \begin{array}{ccc} z & x & y \end{array} \\ \begin{pmatrix} m_1 & 0 & 0 \\ m_2 & m_1 & 0 \\ m_3 & 0 & m_1 \\ 0 & m_2 & 0 \\ 0 & m_3 & m_2 \\ 0 & 0 & m_3 \end{pmatrix} \end{array}. \tag{6.104}$$

The matrix M_{IJ} is a 6×6 matrix, with every entry in (6.104) representing a matrix block of dimension 1×2. Let us, for example, find the $z^2 - z$ component of (6.104) where

$$H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} - 3F))|_{b^{(1)}_{-3}} \overset{m_1}{\to} H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(6\sigma|_{\mathbb{P}^1\times T^2} - 2F))|_{c^{(1)}_{-1}}\,. \tag{6.105}$$

Now, we relate data on $\mathbb{P}^1 \times T^2$ to data on the curve $\mathcal{S}$. Using equations (6.93)-(6.96), we can identify

$$H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} - 3F))|_{b^{(1)}_{-3}} = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1))^* \tag{6.106}$$

and

$$H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(6\sigma|_{\mathbb{P}^1\times T^2} - 2F))|_{c^{(1)}_{-2}} = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}})^*\,. \tag{6.107}$$

In terms of the two dimensional linear space (6.29), we see that

$$H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} - 3F))|_{b^{(1)}_{-3}} = \hat{V}^* \tag{6.108}$$

and

$$H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(6\sigma|_{\mathbb{P}^1\times T^2} - 2F))|_{c^{(1)}_{-2}} = \mathbb{C}\,. \tag{6.109}$$

Similarly, it follows from (6.97) and (6.29) that m_1 is an element of

$$H^0(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} + F))|_{m_1} = \hat{V}\,. \tag{6.110}$$

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Now consider the basis' defined in (6.33), (6.34) and (6.35). Then, any section m_1 can be written as

$$m_1 = \xi_1 u + \xi_2 v\,, \tag{6.111}$$

where $\xi_a, a = 1, 2$, represent the associated moduli. Using the multiplication rules (6.35), we find the explicit 1×2 matrix representation of m_1 in this basis, that is, the M_{11} submatrix of M_{IJ} is given by

$$\begin{array}{cc} & \begin{array}{cc} u^* & v^* \end{array} \\ 1 & \begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix} \end{array}. \tag{6.112}$$

Continuing in this way, we can construct the entire matrix M_{IJ}. The answer is

$$M_{IJ} = \begin{pmatrix} \xi_1 & \xi_2 & 0 & 0 & 0 & 0 \\ \tau_1 & \tau_2 & \xi_1 & \xi_2 & 0 & 0 \\ \zeta_1 & \zeta_2 & 0 & 0 & \xi_1 & \xi_2 \\ 0 & 0 & \tau_1 & \tau_2 & 0 & 0 \\ 0 & 0 & \zeta_1 & \zeta_2 & \tau_1 & \tau_2 \\ 0 & 0 & 0 & 0 & \zeta_1 & \zeta_2 \end{pmatrix}, \tag{6.113}$$

where we have used the following parametrization of m_2 and m_3 in terms of the moduli $\tau_b, b = 1, 2$, and $\zeta_c, c = 1, 2$

$$\begin{aligned} m_2 &= \tau_1 u + \tau_2 v\,, \\ m_3 &= \zeta_1 u + \zeta_2 v\,. \end{aligned} \tag{6.114}$$

Surprisingly, the determinant of the matrix (6.113) turns out to be identically zero for all values of the moduli. We conclude the following.

- The determinant of f_C is given by

  $$\det f_C = 0 \tag{6.115}$$

  for all values of the moduli.

Therefore, in this example, unlike in the previous two cases, $\det f_C$ vanishes identically. Note that the data (6.90) and (6.91) does not satisfy the first condition in (5.46). Therefore, in the present example, the conditions (5.46) are not responsible for the vanishing of $\det f_C$.

As a fourth example, consider

Example 4. Here we take

$$r = 2\,, \qquad n = 3\,, \qquad a > 5 \tag{6.116}$$

and, hence, $\pi^* \mathcal{S} = \mathbb{P}^1 \times T^2$. The vector bundle on this surface is specified by

$$b - 2a = 4\,, \qquad \lambda = \frac{3}{2}\,. \tag{6.117}$$

Recall from the previous section that the cohomology exact sequence under consideration is

$$0 \to H^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to W_5 \stackrel{f_C}{\to} W_6 \to H^1(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to 0\,, \tag{6.118}$$

where

$$W_5 = H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1 \times T^2}(3\sigma|_{\mathbb{P}^1 \times T^2} - 5F)) \tag{6.119}$$

and

$$W_6 = H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1 \times T^2}(6\sigma|_{\mathbb{P}^1 \times T^2} - 3F))\,. \tag{6.120}$$

The relation of W_5 and W_6 to the cohomology groups on $\mathcal{S}$ is the following

$$W_5 \simeq H^1\left(\mathcal{S}, \bigoplus_{i=1}^{3} \mathcal{O}_{\mathcal{S}}(-5)\right) \simeq H^0\left(\mathcal{S}, \bigoplus_{1}^{3} \mathcal{O}_{\mathcal{S}}(3)\right)^* \tag{6.121}$$

and

$$W_6 \simeq H^1\left(\mathcal{S}, \bigoplus_{i=1}^{6} \mathcal{O}_{\mathcal{S}}(-3)\right) \simeq H^0\left(\mathcal{S}, \bigoplus_{i=1}^{6} \mathcal{O}_{\mathcal{S}}(1)\right)^*\,. \tag{6.122}$$

The dimension of each of the linear spaces W_5 and W_6 is 12 and, hence, the linear map f_C can be represented by a 12×12 matrix. This matrix depends on the parameters of the spectral cover restricted to $\mathbb{P}^1 \times T^2$, that is, on the nine homogeneous coordinates of the projective space

$$\mathbb{P}H^0(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1 T^2}(3\sigma|_{\mathbb{P}^1 \times T^2} + 2F)) \simeq \mathbb{P}H^0\left(\mathcal{S}, \bigoplus_{i=1}^{3} \mathcal{O}_{\mathcal{S}}(2)\right) \simeq \mathbb{P}^8\,. \tag{6.123}$$

An arbitrary element $\tilde{w}_5 \in H^1(\mathcal{S}, \bigoplus_{i=1}^3 \mathcal{O}_{\mathcal{S}}(-5))$ can be written as

$$\tilde{w}_5 = B_{-5}^{(1)} \oplus B_{-5}^{(2)} \oplus B_{-5}^{(3)}\,, \tag{6.124}$$

where $B_{-5}^{(i)}, i = 1, 2, 3$, are elements of the four-dimensional vector space $H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-5))$. Let $b_{-5}^{(i)} = \pi^* B_{-5}^{(i)}$ be the lift of $B_{-5}^{(i)}$. We can now write any element $w_5 \in H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1 \times T^2}(3\sigma|_{\mathbb{P}^1 \times T^2} - 5F))$ as

$$w_5 = b_{-5}^{(1)} z + b_{-5}^{(2)} x + b_{-5}^{(3)} y. \tag{6.125}$$

In a similar way, any element $w_6 \in H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1 \times T^2}(6\sigma|_{\mathbb{P}^1 \times T^2} - 3F))$ can be expressed as

$$w_6 = c_{-3}^{(1)} z^2 + c_{-3}^{(2)} xz + c_{-3}^{(3)} yz + c_{-3}^{(4)} x^2 + c_{-3}^{(5)} xy + c_{-3}^{(6)} y^2\,, \tag{6.126}$$

where for $j = 1, \ldots, 6, c_{-3}^{(j)} = \pi^* C_{-3}^{(j)}$ are elements of the two-dimensional vector space $H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1 \times T^2}(-3F))$ and $C_{-3}^{(j)}$ are elements of $H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-3))$. We now need to express the map f_C in terms of the data on the curve $\mathcal{S}$. By using (6.123), f_C can be written as

$$f_C = m_1 z + m_2 x + m_3 y\,, \tag{6.127}$$

where $m_k = \pi^* M_k, k = 1,2,3$, are elements of $H^0(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(2F))$ and M_k are elements of the three-dimensional vector space $H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(2))$. Although there are nine parameters m_k, they are homogeneous coordinates for the eight-dimensional projective space $\mathbb{P}H^0(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} + 2F))$.

Putting this all together, we can specify the linear mapping $W_5 \stackrel{f_C}{\to} W_6$. With respect to fixed basis vectors of W_5 and W_6 the linear map f_C is a 12×12 matrix. To find this matrix, we have to study its action on these basis vectors. The action is generated through multiplication by a section f_C of the form (6.127). Suppressing, for the time being, the vector coefficients $b^{(i)}_{-5}$ and $c^{(i)}_{-3}$ we see from (6.125) that the linear space W_5 is spanned by the following basis vector blocks

$$z\,, \qquad x\,, \qquad y \tag{6.128}$$

whereas it follows from (6.126) that the linear space W_6 is spanned by basis vector blocks

$$z^2\,, \qquad xz\,, \qquad yz\,, \qquad x^2\,, \qquad xy\,, \qquad y^2\,. \tag{6.129}$$

The explicit matrix M_{IJ} is obtained by multiplying the basis vector blocks (6.128) by f_C in (6.127). Expanding the resulting vectors in W_6 in the basis (6.129) yields the matrix. We find that M_{IJ} is given by

$$\begin{array}{c c} & \begin{array}{ccc} z & x & y \end{array} \\ \begin{array}{c} z^2 \\ xz \\ yz \\ x^2 \\ xy \\ y^2 \end{array} & \begin{pmatrix} m_1 & 0 & 0 \\ m_2 & m_1 & 0 \\ m_3 & 0 & m_1 \\ 0 & m_2 & 0 \\ 0 & m_3 & m_2 \\ 0 & 0 & m_3 \end{pmatrix} \end{array}. \tag{6.130}$$

The matrix M_{IJ} is a 12×12 matrix with every entry in (6.130) representing a matrix block of dimension 2×4. Let us, for example, find the $z^2 - z$ component of (6.130) where

$$H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} - 5F))|_{b^{(1)}_{-5}} \stackrel{m_1}{\to} H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(6\sigma|_{\mathbb{P}^1\times T^2} - 3F))|_{c^{(1)}_{-3}}\,. \tag{6.131}$$

Now we relate data on $\mathbb{P}^1 \times T^2$ to data on curve $\mathcal{S}$. Using equations (6.119)-(6.122), we can identify

$$H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} - 5F))|_{b^{(1)}_{-5}} = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(3))^* \tag{6.132}$$

and

$$H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(6\sigma|_{\mathbb{P}^1\times T^2} - 3F))|_{c^{(1)}_{-3}} = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1))^*\,. \tag{6.133}$$

Defining the two dimensional linear space (6.29), we see that

$$H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} - 5F))|_{b^{(1)}_{-5}} = \mathrm{Sym}^3\,\hat{V}^* \tag{6.134}$$

and

$$H^1(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(6\sigma|_{\mathbb{P}^1\times T^2} - 3F))|_{c^{(1)}_{-3}} = \hat{V}^*\,. \tag{6.135}$$

Similarly, it follows from (6.123), (6.29) that m_1 is an element of

$$H^0(\mathbb{P}^1 \times T^2, \mathcal{O}_{\mathbb{P}^1\times T^2}(3\sigma|_{\mathbb{P}^1\times T^2} + 2F))|_{m_1} = \mathrm{Sym}^2\, \hat{V}\,. \tag{6.136}$$

Now consider the basis' defined in (6.33), (6.34) and (6.35). Then

$$\{u^{*3}, u^{*2}v^*, u^*v^{*2}, v^{*3}\} \in \mathrm{Sym}^3\, \hat{V}^* \tag{6.137}$$

is a basis for $\mathrm{Sym}^3\, \hat{V}^*$ and

$$\{u^2, uv, v^2\} \in \mathrm{Sym}^2\, \hat{V}\,. \tag{6.138}$$

Furthermore, any section m_1 can be written as

$$m_1 = \alpha_1 u^2 + \alpha_2 uv + \alpha_3 v^2\,, \tag{6.139}$$

where $\alpha_a, a = 1, 2, 3$, represent the associated moduli. Using the multiplication rules (6.35), we can find the explicit 2×4 matrix representation of m_1 in this basis, that is, the M_{11} submatrix of M_{IJ}. It is given by

$$\begin{array}{c} \\ u^* \\ v^* \end{array} \begin{array}{c} \begin{array}{cccc} u^{*3} & u^{*2}v^* & u^*v^{*2} & v^{*3} \end{array} \\ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \end{array}. \tag{6.140}$$

Continuing in this way, we can construct the entire matrix M_{IJ}. The answer is

$$M_{IJ} = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta_1 & \beta_2 & \beta_3 & 0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & 0 & 0 & 0 \\
0 & \beta_1 & \beta_2 & \beta_3 & 0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & 0 & 0 \\
\gamma_1 & \gamma_2 & \gamma_3 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 \\
0 & \gamma_1 & \gamma_2 & \gamma_3 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 \\
0 & 0 & 0 & 0 & \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 & 0 & \beta_1 & \beta_2 & \beta_3 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 & 0 & \beta_1 & \beta_2 & \beta_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3
\end{pmatrix}. \tag{6.141}$$

where we have used the following parametrization of m_2 and m_3 in terms of the moduli $\beta_b, b = 1, 2, 3$, and $\gamma_c, c = 1, 2, 3$

$$\begin{aligned} m_2 &= \beta_1 u^2 + \beta_2 uv + \beta_3 v^2\,, \\ m_3 &= \gamma_1 u^2 + \gamma_2 uv + \gamma_3 v^2\,. \end{aligned} \tag{6.142}$$

We now can compute the determinant and find the following.

- The determinant of the matrix M_{IJ} is given by

$$\det f_C = -\mathcal{R}^4 \,, \tag{6.143}$$

where

$$\mathcal{R} = \alpha_1\beta_2\gamma_3 - \alpha_1\beta_3\gamma_2 + \alpha_2\beta_3\gamma_1 - \alpha_2\beta_1\gamma_3 + \alpha_3\beta_1\gamma_2 - \alpha_3\beta_2\gamma_1 \,. \tag{6.144}$$

We conclude this example by pointing out that $\det f_C$ will vanish precisely on the zero locus of the homogeneous polynomial $\mathcal{R}$. Globally, $\mathcal{R}$ must be a holomorphic section of some complex line bundle over $\mathbb{P}^8$. Therefore, there must exist a divisor $D_{\mathcal{R}} \subset \mathbb{P}^8$ such that $\mathcal{R}$ is a section of

$$\mathcal{O}_{\mathbb{P}^8}(D_{\mathcal{R}}) \tag{6.145}$$

and vanishes on the co-dimension one submanifold $D_{\mathcal{R}}$ of $\mathbb{P}^8$. This categorizes the zeroes of $\mathcal{R}$ and, hence, $\det f_C$. The exact co-dimension one submanifold

$$D_{\mathcal{R}} \subset \mathbb{P}H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(2) \oplus \mathcal{O}_{\mathcal{S}}(2) \oplus \mathcal{O}_{\mathcal{S}}(2)) \tag{6.146}$$

can be determined by solving the equation $\mathcal{R} = 0$ using (6.144).

7. Expressions for the superpotential

In the previous section, we computed, for several non-trivial examples, the explicit expressions for $\det f_C$ in terms of the associated vector bundle moduli. In each case, $h^0(z, V|_z(-1))$ will be non-zero and, hence, the pfaffian will vanish if and only if

$$\det f_C = 0 \,. \tag{7.1}$$

In special cases, such as those described by conditions (5.46) or Example 3, $\det f_C$ may vanish everywhere in moduli space. In these cases, the conclusions are straightforward. The pfaffian and, hence, the superpotential W vanish for all values of the moduli. That is,

$$W = 0 \,. \tag{7.2}$$

Generically, however, $\det f_C$ will not vanish, as was shown explicitly in examples 2, 1 and 4. However, even in this case, there will always be a divisor D of the vector bundle moduli space where (7.1) is satisfied. For examples 2, 1 and 4, this divisor is given in equations (6.44), (6.89) and (6.146) respectively. Therefore, on D, $\det f_C$ and, hence, the pfaffian vanish. This will allow us to explicitly compute the pfaffian.

To proceed, note that the pfaffian of the chiral Dirac operator in $8k + 2$ dimensions also represents a global section of a line bundle over the space of parameters (see [35] for a review). That it is a section, rather than a function, is a reflection of the fact that the pfaffian is not gauge invariant. In our case, it is a global section of a line bundle over $\mathbb{P}^N$. Any global section of a line bundle over $\mathbb{P}^N$ is known to be a homogeneous polynomial.

This means that the pfaffian must be a polynomial. Furthemore, we have shown that this polynomial vanishes precisely along the same zero locus as $\det f_C$. Therefore, since $\mathbb{P}^N$ is compact, the pfaffian and $\det f_C$ must be equal to each other up to a possible power. One can define a purely algebraic analogue of $\mathrm{Pfaff}(\mathcal{D}_-)$. A more careful analysis of our argument shows that, quite generically, the algebraic version of $\mathrm{Pfaff}(\mathcal{D}_-)$ equals $\det f_C$, up to a constant. Furthermore, it was shown in [36, 37, 38] that the algebraic and analytic notions of $\mathrm{Pfaff}(\mathcal{D}_-)$ agree. Therefore,

$$\mathrm{Pfaff}(\mathcal{D}_-) \propto \det f_C\,. \tag{7.3}$$

One can now present the final answer for the vector bundle moduli contribution to the non-perturbative superpotential. Since the superpotential is proportional to the pfaffian, we conclude that

$$W \propto \det f_C\,. \tag{7.4}$$

Let us now illustrate this explicitly for the three examples considered above. The ordering of the examples will be the same as in the previous subsection.

Example 2. In this example, $\det f_C$ is a polynomial of degree 20, so it is a global section of the line bundle of degree 20 over $\mathbb{P}^{12}$.

- The superpotential equals

  $$W \propto \mathcal{P}^4\,, \tag{7.5}$$

  where

  $$\begin{aligned}\mathcal{P} = {} & \chi_1^2\chi_3\phi_3^2 - \chi_1^2\chi_2\phi_3\phi_4 - 2\chi_1\chi_3^2\phi_3\phi_1 - \\ & - \chi_1\chi_2\chi_3\phi_3\phi_2 + \chi_2^2\chi_3\phi_1\phi_3 + \phi_4^2\chi_1^3 - \\ & - 2\phi_2\phi_4\chi_3\chi_1^2 + \chi_1\chi_3^2\phi_2^2 + 3\phi_1\phi_4\chi_1\chi_2\chi_3 + \\ & + \phi_2\chi_1\phi_4\chi_2^2 + \phi_1^2\chi_3^3 - \phi_2\chi_2\phi_1\chi_3^2 - \phi_4\phi_1\chi_2^3\,.\end{aligned} \tag{7.6}$$

Note that W depends on only seven of the thirteen projective vector bundle moduli which parameterize $\mathbb{P}^{12}$.

Example 1. In this example, $\det f_C$ is a polynomial of degree 44 on the projective space $\mathbb{P}^9$.

- The superpotential is given by

  $$W \propto \mathcal{Q}^{11}\tilde{\mathcal{Q}}\,, \tag{7.7}$$

  where

  $$\mathcal{Q} = \rho_1\rho_2\lambda_2 - \lambda_1\rho_2^2 - \lambda_3\rho_1^2 \tag{7.8}$$

  and $\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}(\lambda_a, \rho_b, \mu_c, h_d)$ is given in the appendix D.

Note that the superpotential depends on all ten projective vector bundle moduli which parameterize $\mathbb{P}^9$ and, in addition, contains non-trivial interactions between vector bundle moduli and some moduli of the underlying Calabi-Yau space.

Example 4. In this example, $\det f_C$ is a polynomial of degree 12 depending on nine homogeneous variables and, therefore, it is a global section of the line bundle of degree 12 over $\mathbb{P}^8$.

- The superpotential is

$$W \propto \mathcal{R}^4 \,, \tag{7.9}$$

where

$$\mathcal{R} = \alpha_1\beta_2\gamma_3 - \alpha_1\beta_3\gamma_2 + \alpha_2\beta_3\gamma_1 - \alpha_2\beta_1\gamma_3 + \alpha_3\beta_1\gamma_2 - \alpha_3\beta_2\gamma_1 \,. \tag{7.10}$$

Note that W depends on all nine projective vector bundle moduli which parameterize $\mathbb{P}^8$. As in example 2, W is given by a perfect power of a polynomial.

A. Evaluation of $\int \mathcal{D}ge^{-S_{WZW0}}$

In this appendix we will evaluate the path integral

$$\int \mathcal{D}ge^{-S_{WZW0}} \,, \tag{A.1}$$

where

$$\begin{aligned} S_{WZW0} = &-\frac{1}{8\pi}\int_z \mathrm{d}\sigma^0\mathrm{d}\sigma^1 \,\mathrm{tr}\left[\frac{1}{2}\sqrt{h}h^{ij}(\omega_i - A_i)(\omega_j - A_j) + i\epsilon^{ij}\omega_i A_j\right] + \\ &+\frac{1}{24}\int_{\mathcal{B}} \mathrm{d}^3\hat{\sigma} i\epsilon^{\hat{i}\hat{j}\hat{k}}\,\mathrm{tr}\,\omega'_{\hat{k}}\omega'_{\hat{j}}\omega'_{\hat{i}} \,, \end{aligned} \tag{A.2}$$

the Lie algebra valued one form ω is given by

$$\omega = g^{-1}dg \tag{A.3}$$

and A_i is the pullback of the hermitean connection on the Calabi-Yau threefold X to the holomorphic curve z. If we introduce the complex coordinates on z

$$\sigma = \sigma^0 + i\sigma^1 \,, \qquad \bar{\sigma} = \sigma^0 - i\sigma^1 \,, \tag{A.4}$$

the action S_{WZW0} can be written as

$$S_{WZW0} = S(g) + \frac{1}{4\pi}\int_z \mathrm{d}^2\sigma \,\mathrm{tr}\,\omega_{\bar{\sigma}}A_\sigma - \frac{1}{8\pi}\int_z \mathrm{d}^2\sigma\,\mathrm{tr}\,A_{\bar{\sigma}}A_\sigma \,, \tag{A.5}$$

where

$$\mathrm{d}^2\sigma = \mathrm{d}\sigma\mathrm{d}\bar{\sigma} \tag{A.6}$$

and

$$S(g) = -\frac{1}{8\pi}\int_z \mathrm{d}^2\sigma\,\mathrm{tr}(g^{-1}\partial_\sigma g)(g^{-1}\partial_{\bar{\sigma}}g) + \frac{1}{24\pi}\int_{\mathcal{B}} \mathrm{d}^3\hat{\sigma} i\epsilon^{\hat{i}\hat{j}\hat{k}}\,\mathrm{tr}(g'^{-1}\partial_{\hat{k}}g')(g'^{-1}\partial_{\hat{j}}g')(g'^{-1}\partial_{\hat{i}}g') \tag{A.7}$$

is the WZW action with zero background gauge field. Since A_i is a connection on a holomorphic vector bundle and, hence, satisfies the hermitean Yang-Mills equations, in any neighborhood $\mathcal{U} \subset z$ it is of the form

$$A_\sigma = \partial_\sigma a \cdot a^{-1}\,, \qquad A_{\bar\sigma} = \partial_{\bar\sigma} a^{\dagger -1} \cdot a^\dagger\,, \tag{A.8}$$

where a is a map from $\mathcal{U}$ to the complexification of the structure group G. Using (A.8) and the Polyakov-Wiegmann identity [39]

$$S(a_1 a_2) = S(a_1) + S(a_2) + \frac{1}{4\pi}\int_z \mathrm{d}^2\sigma\,\mathrm{tr}(a_1^{-1}\partial_{\bar\sigma} a_1)(a_2 \partial_\sigma a_2^{-1})\,, \tag{A.9}$$

the action (A.5) can be re-written as

$$S_{WZW0} = S(ga) - S(a) - \frac{1}{8\pi}\int_z \mathrm{d}^2\sigma(\partial_\sigma a \cdot a^{-1})(\partial_{\bar\sigma} a^{\dagger -1}\cdot a^\dagger)\,. \tag{A.10}$$

It is now easy to evaluate the integral (A.1). Since $\mathcal{D}g$ is the right-invariant group measure, the integral

$$\int \mathcal{D}g e^{-S(ga)} \tag{A.11}$$

is a constant, independent of a and, hence, of the vector bundle moduli. Therefore, we have the following result for the integral (A.1)

$$\int \mathcal{D}g e^{-S_{WZW0}} \propto exp\left(S(a) + \frac{1}{8\pi}\int_z \mathrm{d}^2\sigma\,\mathrm{tr}(\partial_\sigma a\cdot a^{-1})(\partial_{\bar\sigma} a^{\dagger -1}\cdot a^\dagger)\right)\,, \tag{A.12}$$

where the functional S is defined in (A.7) with g replaced by a and the relation between a and the gauge connection A is given in (A.8). Note that the result is expressed in terms of a rather than the gauge connection A. It does not appear to be possible to present (A.12) soley in terms of the gauge connection. Also, note that a is globally defined since A is.

Equation (A.12) represents the general formula for the vector bundle moduli superpotential and, also, for the pfaffian of the chiral Dirac operator twisted by a holomorphic vector bundle whose structure group is contained in SO(16) $\times$ SO(16). To find the explicit result, one has to find an exact solution of the hermitean Yang-Mills equations on a Calabi-Yau threefold together with the integration constants, pull it back to the curve z and perform the integration over the curve. As we discussed in section 2, this procedure is problematic since no solutions of the hermitean Yang-Mills equations on a Calabi-Yau threefold are known.

It is straightforward to check that the expression in the exponent in (A.12) transforms under a gauge transformation with parameter ϵ as

$$-\frac{i}{8\pi}\int_z \mathrm{d}^2\sigma\,\mathrm{tr}(\epsilon dA)\,. \tag{A.13}$$

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This anomaly is cancelled by the variation of the B-field [5]

$$\delta B = \frac{1}{8\pi}\,\mathrm{tr}(\epsilon dA)\,, \tag{A.14}$$

leaving the superpotential (2.17) gauge invariant.[1] Note that expression (A.13) precisely equals the anomaly of the square root of the chiral Dirac operator.

B. Spectral data and $\mathrm{SU}(n)$ vector bundles

B.1 Elliptically fibered Calabi-Yau threefolds

In this appendix, we will consider Calabi-Yau threefolds, X, that are structured as elliptic curves fibered over a base surface, B. We denote the natural projection as $\pi : X \to B$ and by $\sigma : B \to X$ the analytic map that defines the zero section.

A simple representation of an elliptic curve is given in the projective space $\mathbb{CP}^2$ by the Weierstrass equation

$$zy^2 = 4x^3 - g_2 xz^2 - g_3 z^3\,, \tag{B.1}$$

where (x, y, z) are the homogeneous coordinates of $\mathbb{CP}^2$ and g_2, g_3 are constants. This same equation can represent the elliptic fibration, X, if the coefficients g_2, g_3 in the Weierstrass equation are functions over the base surface, B. The correct way to express this globally is to replace the projective plane $\mathbb{CP}^2$ by a $\mathbb{CP}^2$-bundle $P \to B$ and then require that g_2, g_3 be sections of appropriate line bundles over the base. If we denote the conormal bundle to the zero section $\sigma(B)$ by $\mathcal{L}$, then $P = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$, where $\mathbb{P}$ stands for the projectivization of the vector bundle. There is a hyperplane line bundle $\mathcal{O}_P(1)$ on P which corresponds to the divisor $\mathbb{P}(\mathcal{L}^2 \oplus \mathcal{L}^3) \subset P$ and the coordinates x, y, z are sections of $\mathcal{O}_P(1) \otimes \mathcal{L}^2, \mathcal{O}_P(1) \otimes \mathcal{L}^3$ and $\mathcal{O}_P(1)$ respectively. It then follows from (B.1) that the coefficients g_2 and g_3 are sections of $\mathcal{L}^4$ and $\mathcal{L}^6$.

It is useful to define new coordinates, $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ on X by $x = \mathbf{XZ}$, $y = \mathbf{Y}$ and $z = \mathbf{Z}^3$. It follows that $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are now sections of line bundles

$$\mathbf{X} \sim \mathcal{O}_X(2\sigma) \otimes \mathcal{L}^2\,, \qquad \mathbf{Y} \sim \mathcal{O}_X(3\sigma) \otimes \mathcal{L}^3\,, \qquad \mathbf{Z} \sim \mathcal{O}_X(\sigma) \tag{B.2}$$

respectively, where we use the fact that $\mathcal{O}_X(3\sigma) = \mathcal{O}_P(1)|_X$. The coefficients g_2 and g_3 remain sections of line bundles

$$g_2 \sim \mathcal{L}^4\,, \qquad g_3 \sim \mathcal{L}^6\,. \tag{B.3}$$

The symbol "$\sim$" simply means "section of".

The requirement that elliptically fibered threefold, X, be a Calabi-Yau space constrains the first Chern class of the tangent bundle, TX, to vanish. That is,

$$c_1(TX) = 0\,. \tag{B.4}$$

[1] The coefficients in (A.13) and (A.14) are different from those in [5] since we are following the conventions of [6, 7] for the normalization of the Wess-Zumino-Witten term.

It follows from this that

$$\mathcal{L} = K_B^{-1}\,, \tag{B.5}$$

where K_B is the canonical bundle on the base, B. Condition (B.5) is rather strong and restricts the allowed base spaces of an elliptically fibered Calabi-Yau threefold to be del Pezzo, Hirzebruch and Enriques surfaces, as well as certain blow-ups of Hirzebruch surfaces [40]. The definitions and properties of these surfaces have been reviewed in [16]. In this paper, we will primarily be concerned with the case

$$B = \mathbb{F}_r \tag{B.6}$$

where r is any non-negative integer. Recall from [16] that the homology group $H_2(\mathbb{F}_r, \mathbb{Z})$ has as a basis of effective classes $\mathcal{S}$ and $\mathcal{E}$ with the intersection numbers

$$\mathcal{S}^2 = -r\,, \qquad \mathcal{S}\cdot\mathcal{E} = 1\,, \qquad \mathcal{E}^2 = 0 \tag{B.7}$$

and that the Chern classes are given by

$$c_1(\mathbb{F}_r) = 2\mathcal{S} + (r+2)\mathcal{E}\,, \qquad c_2(\mathbb{F}_r) = 4\,. \tag{B.8}$$

B.2 Spectral cover description of SU(n) vector bundles

As discussed in [34, 41], SU(n) vector bundles over an elliptically fibered Calabi-Yau threefold can be explicitly constructed from two mathematical objects, a divisor $\mathcal{C}$ of X, called the spectral cover, and a line bundle $\mathcal{N}$ on $\mathcal{C}$. In this paper, we will consider only stable SU(n) vector bundles constructed from irreducible spectral covers. In addition, we will impose the condition that the spectral cover be a "positive" divisor.

B.3 Spectral cover

A spectral cover, $\mathcal{C}$, is a surface in X that is an n-fold cover of the base B. That is, $\pi_{\mathcal{C}} : \mathcal{C} \to B$. The general form for a spectral cover is given by

$$\mathcal{C} = n\sigma + \pi^*\eta\,, \tag{B.9}$$

where σ is the zero section and η is some curve in the base B. The terms in (B.9) can be considered either as elements of the homology group $H_4(X,\mathbb{Z})$ or, by Poincare duality, as elements of cohomology $H^2(X,\mathbb{Z})$.

In terms of the coordinates $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ introduced above, it can be shown that the spectral cover can be represented as the zero set of the polynomial

$$s = a_0\mathbf{Z}^n + a_2\mathbf{X}\mathbf{Z}^{n-2} + a_3\mathbf{Y}\mathbf{Z}^{n-3} + \cdots + a_n\mathbf{X}^{\frac{n}{2}} \tag{B.10}$$

for n even and ending in $a_n\mathbf{X}^{\frac{n-3}{2}}\mathbf{Y}$ if n is odd, along with the relations (B.2). This tells us that the polynomial s must be a holomorphic section of the line bundle of the spectral cover, $\mathcal{O}_X(\mathcal{C})$. That is,

$$s \sim \mathcal{O}_X(n\sigma + \pi^*\eta)\,. \tag{B.11}$$

It follows from this and equation (B.2), that the coefficients a_i in the polynomial s must be sections of the line bundles

$$a_i \sim \pi^* K_B^i \otimes \mathcal{O}_X(\pi^* \eta) \tag{B.12}$$

for $i = 1, \ldots, n$ where we have used expression (B.5).

In order to describe vector bundles most simply, there are two properties that we require the spectral cover to possess. The first, which is shared by all spectral covers, is that

- $\mathcal{C}$ must be an effective class in $H_4(X, \mathbb{Z})$.

This property is simply an expression of the fact the spectral cover must be an actual surface in X. It can easily be shown that

$$\mathcal{C} \subset X \text{ is effective } \iff \eta \text{ is an effective class in } H_2(B, \mathbb{Z})\,. \tag{B.13}$$

The second property that we require for the spectral cover is that

- $\mathcal{C}$ is an irreducible surface.

This condition is imposed because it guarantees that the associated vector bundle is stable.

In this paper, we require a third condition on the spectral cover, namely that

- $\mathcal{C}$ is positive.

This condition is imposed because it allows one to use the Kodaira vanishing theorem and the Lefschetz theorem when evaluating the number of vector bundle moduli. These theorems greatly simplify this calculation. By definition, $\mathcal{C}$ is positive (or ample) if and only if the first Chern class of the associated line bundle $\mathcal{O}_X(\mathcal{C})$ is a positive class in $H^2_{DR}(X, \mathbb{R})$. An equivalent definition of positive is the following. $\mathcal{C}$ is a positive divisor if and only if

$$\mathcal{C} \cdot c > 0 \tag{B.14}$$

for every effective curve c in X.

In order to make these concepts more concrete, we take the base surface to be

$$B = \mathbb{F}_r \tag{B.15}$$

and write the conditions under which the three properties stated above are satisfied. Then, in general, $\mathcal{C}$ is given by expression (B.9) where

$$\eta = a\mathcal{S} + b\mathcal{E} \tag{B.16}$$

and a,b are integers. One can easily check that η is an effective class in $\mathbb{F}_r$ and, hence, that $\mathcal{C}$ is an effective class in X, if and only if

$$a \geq 0, \qquad b \geq 0\,. \tag{B.17}$$

It was shown in [22] that $\mathcal{C}$ is irreducible if and only if

$$b \geq ar \tag{B.18}$$

and

$$a \geq 2n, \qquad b \geq n(r+2)\,. \tag{B.19}$$

Finally, it was shown in [11] using expressions (B.7) and (B.8) that spectral cover $\mathcal{C}$ will be positive if and only if

$$a > 2n, \qquad b > ar - n(r-2)\,. \tag{B.20}$$

The line bundle $\mathcal{N}$

As discussed in [34, 41], in addition to the spectral cover it is necessary to specify a line bundle, $\mathcal{N}$, over $\mathcal{C}$. For SU(n) vector bundles, this line bundle must be a restriction of a global line bundle on X (which we will again denote by $\mathcal{N}$), satisfying the condition

$$c_1(\mathcal{N}) = n\left(\frac{1}{2}+\lambda\right)\sigma + \left(\frac{1}{2}-\lambda\right)\pi^*\eta + \left(\frac{1}{2}+n\lambda\right)\pi^*c_1(B)\,, \tag{B.21}$$

where $c_1(\mathcal{N})$, $c_1(B)$ are the first Chern classes of $\mathcal{N}$ and B respectively and λ is, a priori, a rational number. Since $c_1(\mathcal{N})$ must be an integer class, it follows that either

$$n \quad \text{is odd}\,, \qquad \lambda = m + \frac{1}{2} \tag{B.22}$$

or

$$n \quad \text{is even}\,, \qquad \lambda = m\,, \qquad \eta = c_1(B) \mod 2\,, \tag{B.23}$$

where $m \in \mathbb{Z}$.

B.4 SU(n) vector bundle

Given a spectral cover, $\mathcal{C}$, and a line bundle, $\mathcal{N}$, satisfying the above properties, one can now uniquely construct an SU(n) vector bundle, V. This can be accomplished in two ways. First, as discussed in [41], the vector bundle can be directly constructed using the associated Poincare bundle, $\mathcal{P}$. The result is that

$$V = \pi_{1*}(\pi_2^*\mathcal{N} \otimes \mathcal{P})\,, \tag{B.24}$$

where π_1 and π_2 are the two projections of the fiber product $X \times_B \mathcal{C}$ onto the two factors X and $\mathcal{C}$. We refer the reader to [41, 42] for a detailed discussion. Equivalently, V can be constructed directly from $\mathcal{C}$ and $\mathcal{N}$ using the Fourier-Mukai transformation, as discussed in [34, 41]. Both of these constructions work in reverse, yielding the spectral data $(\mathcal{C}, \mathcal{N})$ up to the overall factor of K_B given the vector bundle V. Throughout this paper we will indicate this relationship between the spectral data and the vector bundle by writing

$$(\mathcal{C}, \mathcal{N}) \longleftrightarrow V\,. \tag{B.25}$$

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The Chern classes for the SU(n) vector bundle V have been computed in [34] and [42, 43]. The results are

$$c_1(V) = 0 \tag{B.26}$$

since $\mathrm{tr}\, F = 0$ for the structure group SU(n),

$$c_2(V) = \pi^*\eta \cdot \sigma - \frac{1}{24}\pi^* c_1(B)^2(n^3 - n) + \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right) n\pi^*(\eta \cdot (\eta - nc_1(B))) \tag{B.27}$$

and

$$c_3(V) = 2\lambda\sigma \cdot \pi^*(\eta \cdot (\eta - nc_1(B)))\,. \tag{B.28}$$

Finally, we note that it was shown in [42] that

$$N_{gen} = \frac{c_3(V)}{2} \tag{B.29}$$

gives the number of quark and lepton generations.

C. The direct image $\pi_* \mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}} + fF)$

The goal of this appendix is to show how to find the direct image of the line bundle

$$\mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}} + fF)\,, \qquad A > 0 \tag{C.1}$$

on the curve $\mathcal{S} \subset \mathbb{F}_r$ used throughout the paper.

First, note that

$$\mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}} + fF) = \mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}}) \otimes \mathcal{O}_{\pi^*\mathcal{S}}(fF) = \mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}}) \otimes \pi^*\mathcal{O}_{\mathcal{S}}(f)\,. \tag{C.2}$$

From the projection formula

$$\pi_*(\mathcal{F} \otimes \pi^*\mathcal{G}) = (\pi_*\mathcal{F}) \otimes \mathcal{G} \tag{C.3}$$

for arbitrary bundles $\mathcal{F}$ and $\mathcal{G}$, it follows that

$$\pi_*\mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}} + fF) = \pi_*\mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}}) \otimes \mathcal{O}_{\mathcal{S}}(f)\,. \tag{C.4}$$

This implies that it is enough to find $\pi_*\mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}})$, which can be done by induction. To begin with, consider the short exact sequence

$$0 \to \mathcal{O}_{\pi^*\mathcal{S}} \to \mathcal{O}_{\pi^*\mathcal{S}}(\sigma|_{\pi^*\mathcal{S}}) \to \mathcal{O}_{\pi^*\mathcal{S}}(\sigma|_{\pi^*\mathcal{S}})|_{\sigma|_{\pi^*\mathcal{S}}} \to 0\,. \tag{C.5}$$

The inclusion

$$\mathcal{O}_{\pi^*\mathcal{S}} \hookrightarrow \mathcal{O}_{\pi^*\mathcal{S}}(\sigma|_{\pi^*\mathcal{S}}) \tag{C.6}$$

induces a sheaf map

$$i : \pi_*\mathcal{O}_{\pi^*\mathcal{S}} \hookrightarrow \pi_*\mathcal{O}_{\pi^*\mathcal{S}}(\sigma|_{\pi^*\mathcal{S}})\,. \tag{C.7}$$

For each point $p \in \mathcal{S}$, consider a map

$$i_p : H^0(F_p, \mathcal{O}_{F_p}) \to H^0(F_p, \mathcal{O}_{F_p}((\sigma|_{\pi^*\mathcal{S}})|_{F_p}))\,, \tag{C.8}$$

where F_p is the elliptic fiber above the point p. Since $\sigma|_{\pi^*\mathcal{S}}$ intersects F at one point, it follows that the degree of the line bundle $\mathcal{O}_{F_p}((\sigma|_{\pi^*\mathcal{S}})|_{F_p})$ is one and, hence, it has a unique global holomorphic section. Therefore, the map i_p is an isomorphism, so

$$\pi_*\mathcal{O}_{\pi^*\mathcal{S}}(\sigma|_{\pi^*\mathcal{S}}) \simeq \pi_*\mathcal{O}_{\pi^*\mathcal{S}} \simeq \mathcal{O}_{\mathcal{S}}\,. \tag{C.9}$$

Thus, we have found the direct image of the line bundle $\mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}})$ for $A = 1$. Now, consider the sequence

$$0 \to \mathcal{O}_{\pi^*\mathcal{S}}(\sigma|_{\pi^*\mathcal{S}}) \to \mathcal{O}_{\pi^*\mathcal{S}}(2\sigma|_{\pi^*\mathcal{S}}) \to \mathcal{O}_{\pi^*\mathcal{S}}(2\sigma|_{\pi^*\mathcal{S}})|_{\sigma|_{\pi^*\mathcal{S}}} \to 0 \tag{C.10}$$

and take its direct image, remembering that the direct image of the first term was already found in (C.9). By using the facts that the last term in the sequence can be written as $\pi^*\mathcal{O}_{\mathcal{S}}(2\sigma|_{\pi^*\mathcal{S}} \cdot \sigma|_{\pi^*\mathcal{S}})|_{\sigma|_{\pi^*\mathcal{S}}}$ and all higher direct images vanish by the Kodaira vanishing theorem, we obtain

$$0 \to \mathcal{O}_{\mathcal{S}} \to \pi_*\mathcal{O}_{\pi^*\mathcal{S}}(2\sigma|_{\pi^*\mathcal{S}}) \to \mathcal{O}_{\mathcal{S}}(2(r-2)) \to 0\,, \tag{C.11}$$

where the intersection property [11]

$$\sigma|_{\pi^*\mathcal{S}} \cdot \sigma|_{\pi^*\mathcal{S}} = -(2-r) \tag{C.12}$$

has been used. Since the extention group

$$Ext(\mathcal{O}_{\mathcal{S}}(2(r-2)), \mathcal{O}_{\mathcal{S}}) \simeq H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(2(r-2))^* \otimes \mathcal{O}_{\mathcal{S}}) \simeq H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(2r-6))^* \tag{C.13}$$

vanishes for $r = 0, 1, 2$, we have

$$\pi_*\mathcal{O}_{\pi^*\mathcal{S}}(2\sigma|_{\pi^*\mathcal{S}}) \simeq \mathcal{O}_{\mathcal{S}} \oplus \mathcal{O}_{\mathcal{S}}(2(r-2))\,. \tag{C.14}$$

Therefore, we have found the direct image of the line bundle $\pi_*\mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}})$ for $A = 2$. Continuing this way we obtain

$$\pi_*\mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}}) \simeq \mathcal{O}_{\mathcal{S}} \oplus \bigoplus_{i=2}^{A} \mathcal{O}_{\mathcal{S}}(i(r-2)) \tag{C.15}$$

for arbitrary $A \geq 1$. Tensoring with $\mathcal{O}_{\mathcal{S}}(f)$, we get

$$\pi_*\mathcal{O}_{\pi^*\mathcal{S}}(A\sigma|_{\pi^*\mathcal{S}} + fF) \simeq \mathcal{O}_{\mathcal{S}}(f) \oplus \bigoplus_{i=2}^{A} \mathcal{O}_{\mathcal{S}}(f + i(r-2)) \tag{C.16}$$

for arbitrary $A \geq 1$.

D. The polynomial $\tilde{\mathcal{Q}}$

Here, we give the expression for the polynomial $\tilde{\mathcal{Q}}$ that arises in example 2 of section 5. We found that

$$
\begin{aligned}
\tilde{\mathcal{Q}} =& \\
& 48\mu_1^2\lambda_2\rho_2^8 - 48\mu_5^2\rho_1^8\lambda_2 - 2\mu_2\rho_1^6\lambda_3^4 + 2\mu_4\rho_2^6\lambda_1^4 - h_2\rho_2^8\lambda_1^3 + h_4\rho_1^8\lambda_3^3 - 24\mu_3\mu_2\rho_1^3\lambda_2\rho_2^5 - \\
& - 72\mu_3\mu_2\rho_1^2\rho_2^6\lambda_1 + 120\mu_3\mu_2\rho_1^4\lambda_3\rho_2^4 + 96\mu_5\mu_1\rho_1^3\rho_2^5\lambda_1 - 96\mu_5\mu_1\rho_1^5\lambda_3\rho_2^3 + 48\mu_3\mu_1\rho_1\rho_2^7\lambda_1 + \\
& + 48\mu_3\mu_1\rho_1^2\lambda_2\rho_2^6 - 144\mu_3\mu_1\rho_1^3\lambda_3\rho_2^5 - 72\mu_4\mu_1\rho_1^2\rho_2^6\lambda_1 - 24\mu_4\mu_1\rho_1^3\lambda_2\rho_2^5 + 120\mu_4\mu_1\rho_1^4\lambda_3\rho_2^4 - \\
& - 120\mu_4\mu_3\rho_1^4\rho_2^4\lambda_1 + 24\mu_4\mu_3\rho_1^5\lambda_2\rho_2^3 + 72\mu_4\mu_3\rho_1^6\lambda_3\rho_2^2 - 168\mu_5\mu_4\rho_1^6\rho_2^2\lambda_1 + 72\mu_5\mu_4\rho_1^7\lambda_2\rho_2 + \\
& + 24\mu_5\mu_4\rho_1^8\lambda_3 - 48\mu_5\mu_3\rho_1^7\lambda_3\rho_2 - 48\mu_5\mu_3\rho_1^6\lambda_2\rho_2^2 + 144\mu_5\mu_3\rho_1^5\rho_2^3\lambda_1 - 2h_5\rho_1^6\lambda_2^3\rho_2^2 + \\
& + 72\mu_5\mu_2\rho_1^6\lambda_3\rho_2^2 - 12h_5\lambda_1\rho_1^6\lambda_2\lambda_3\rho_2^2 + 4h_5\rho_1^3\rho_2^5\lambda_1^3 + 24\mu_5\mu_2\rho_1^5\lambda_2\rho_2^3 - 24\mu_2\mu_1\rho_2^8\lambda_1 - \\
& - 120\mu_5\mu_2\rho_1^4\rho_2^4\lambda_1 - 2h_5\rho_1^8\lambda_3^2\lambda_2 - 3h_4\rho_1^2\rho_2^6\lambda_1^3 - 72\mu_2\mu_1\lambda_2\rho_1\rho_2^7 + 8h_5\lambda_1^2\rho_1^5\lambda_3\rho_2^3 - \\
& - 10h_5\lambda_1^2\rho_1^4\lambda_2\rho_2^4 - 96\mu_4\mu_2\rho_1^5\lambda_3\rho_2^3 + 96\mu_4\mu_2\rho_1^3\rho_2^5\lambda_1 + 168\mu_2\mu_1\rho_1^2\lambda_3\rho_2^6 + h_4\rho_1^5\lambda_2^3\rho_2^3 + \\
& + 2h_3\rho_1\rho_2^7\lambda_1^3 - 2h_3\rho_1^7\lambda_3^3\rho_2 - 5h_4\lambda_1^2\rho_1^4\lambda_3\rho_2^4 + 7h_4\lambda_1^2\rho_1^3\lambda_2\rho_2^5 + 4h_5\rho_1^7\lambda_3\lambda_2^2\rho_2 + \\
& + 4h_5\lambda_1\rho_1^7\lambda_3^2\rho_2 + 8h_5\lambda_1\rho_1^5\lambda_2^2\rho_2^3 - 2h_3\rho_1^5\lambda_3\lambda_2^2\rho_2^3 - 2h_3\lambda_1\rho_1^5\lambda_3^2\rho_2^3 + 6h_4\lambda_1\rho_1^5\lambda_2\lambda_3\rho_2^3 + \\
& + 2h_3\lambda_1\rho_1^3\lambda_2^2\rho_2^5 + 2h_3\lambda_1^2\rho_1^3\lambda_3\rho_2^5 - 4h_3\lambda_1^2\rho_1^2\lambda_2\rho_2^6 - h_4\rho_1^7\lambda_3^2\lambda_2\rho_2 - h_4\rho_1^6\lambda_3\lambda_2^2\rho_2^2 - \\
& - h_4\lambda_1\rho_1^6\lambda_3^2\rho_2^2 - 5h_4\lambda_1\rho_1^4\lambda_2^2\rho_2^4 + 2h_1\rho_1^2\lambda_2^3\rho_2^6 + 2h_1\lambda_1^2\lambda_2\rho_2^8 + 3h_2\rho_1^6\lambda_3^3\rho_2^2 - \\
& - h_2\rho_1^3\lambda_2^3\rho_2^5 + 4h_3\rho_1^6\lambda_3^2\lambda_2\rho_2^2 - 4h_1\rho_1^5\lambda_3^3\rho_2^3 + 5h_2\lambda_1\rho_1^4\lambda_3^2\rho_2^4 + h_2\lambda_1\rho_1^2\lambda_2^2\rho_2^6 + \\
& + h_2\lambda_1^2\rho_1^2\lambda_3\rho_2^6 + h_2\lambda_1^2\lambda_2\rho_1\rho_2^7 - 6h_2\lambda_1\rho_1^3\lambda_2\lambda_3\rho_2^5 - 2\mu_5\rho_1^5\lambda_2^4\rho_2 - 2\mu_3\lambda_1^3\lambda_2\rho_2^6 - \\
& - 2\mu_4\rho_1^6\lambda_3^2\lambda_2^2 + 2\mu_5\rho_1^6\lambda_3\lambda_2^3 - 8\mu_5\rho_1\rho_2^5\lambda_1^4 + 5h_2\rho_1^4\lambda_3\lambda_2^2\rho_2^4 + 2\mu_3\rho_1^6\lambda_3^3\lambda_2 - \\
& - 8h_1\rho_1^3\lambda_3\lambda_2^2\rho_2^5 - 8h_1\lambda_1\rho_1^3\lambda_3^2\rho_2^5 - 4\mu_5\lambda_1\rho_1^5\lambda_3\lambda_2^2\rho_2 - 4h_1\lambda_1\lambda_2^2\rho_1\rho_2^7 - 4h_1\lambda_1^2\rho_1\lambda_3\rho_2^7 + \\
& + 12h_1\lambda_1\rho_1^2\lambda_3\lambda_2\rho_2^6 - 7h_2\rho_1^5\lambda_3^2\lambda_2\rho_2^3 + 2\mu_4\lambda_1\rho_1^6\lambda_3^3 - 10\mu_4\lambda_1\rho_1^4\lambda_3\lambda_2^2\rho_2^2 + 20\mu_5\rho_1^2\lambda_2\rho_2^4\lambda_1^3 + \\
& + 20\mu_4\lambda_1^2\rho_1^3\lambda_2\lambda_3\rho_2^3 + 10h_1\rho_1^4\lambda_3^2\lambda_2\rho_2^4 - 2\mu_2\lambda_3\rho_2^6\lambda_1^3 + 2\mu_2\lambda_1^2\lambda_2^2\rho_2^6 + 2\mu_3\lambda_1^2\lambda_2^2\rho_1\rho_2^5 + \\
& + 8\mu_3\lambda_1^3\rho_1\lambda_3\rho_2^5 - 2\mu_1\lambda_1\lambda_2^3\rho_2^6 + 10\mu_3\lambda_1\rho_1^4\lambda_3^2\lambda_2\rho_2^2 + 2\mu_4\rho_1^5\lambda_3\lambda_2^3\rho_2 - 10\mu_4\lambda_1^2\rho_1^4\lambda_3^2\rho_2^2 - \\
& - 10\mu_4\lambda_1^3\rho_1^2\lambda_3\rho_2^4 - 10\mu_3\lambda_1^2\rho_1^2\lambda_3\lambda_2\rho_2^4 - 2\mu_4\lambda_1^3\lambda_2\rho_1\rho_2^5 - 4\mu_5\lambda_1\rho_1^6\lambda_3^2\lambda_2 + 6\mu_4\lambda_1\rho_1^5\lambda_3^2\lambda_2\rho_2 + \\
& + 10\mu_5\lambda_1\rho_1^4\lambda_2^3\rho_2^2 + 8\mu_5\lambda_1^2\rho_1^5\lambda_3^2\rho_2 - 20\mu_5\lambda_1^2\rho_1^3\lambda_2^2\rho_2^3 + 2\mu_1\rho_1\lambda_2^4\rho_2^5 + 8\mu_1\rho_1^5\lambda_3^4\rho_2 - \\
& - 6\mu_2\lambda_1^2\rho_1\lambda_2\lambda_3\rho_2^5 - 2\mu_3\rho_1^5\lambda_3^2\lambda_2^2\rho_2 - 8\mu_3\lambda_1\rho_1^5\lambda_3^3\rho_2 + 96\mu_5^2\rho_1^7\rho_2\lambda_1 + 10\mu_2\lambda_1^2\rho_1^2\lambda_3^2\rho_2^4 + \\
& + 72\mu_4^2\rho_1^5\rho_2^3\lambda_1 - 24\mu_4^2\rho_1^7\lambda_3\rho_2 + 4\mu_1\lambda_1^2\lambda_2\lambda_3\rho_2^6 + 2\mu_2\rho_1^5\lambda_2\lambda_3^3\rho_2 + 10\mu_2\lambda_1\rho_1^2\lambda_2^2\lambda_3\rho_2^4 + \\
& + 10\mu_2\lambda_1\rho_1^4\lambda_3^3\rho_2^2 - 2\mu_2\lambda_1\rho_1\lambda_2^3\rho_2^5 - 8\mu_1\lambda_1^2\rho_1\lambda_3^2\rho_2^5 - 20\mu_2\lambda_1\rho_1^3\lambda_3^2\lambda_2\rho_2^3 + 24\mu_2^2\rho_1\rho_2^7\lambda_1 + \\
& + 20\mu_1\rho_1^3\lambda_2^2\lambda_3^2\rho_2^3 - 10\mu_1\rho_1^2\lambda_2^3\lambda_3\rho_2^4 + 24\mu_2^2\rho_1^2\lambda_2\rho_2^6 - 72\mu_2^2\rho_1^3\lambda_3\rho_2^5 + 48\mu_3^2\rho_1^3\rho_2^5\lambda_1 - \\
& - 24\mu_4^2\rho_1^6\lambda_2\rho_2^2 - 96\mu_1^2\rho_1\lambda_3\rho_2^7 - 20\mu_1\rho_1^4\lambda_2\lambda_3^3\rho_2^2 - 48\mu_3^2\rho_1^5\lambda_3\rho_2^3 + 4\mu_1\lambda_1\rho_1\lambda_2^2\lambda_3\rho_2^5\,.
\end{aligned}
\tag{D.1}
$$

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